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Hydromagnetic Conditions near the Core-Mantle Boundary

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1. GENERAL SUMMARY

This grant originally ran from 3/15/92 to 3/14/93 and was renewed for an additional 12 months. NASA accepted a no-cost extension for another year so that money remaining in the grant could be used to support Andrew Walker, the PI's graduate student, whose thesis involves work under the grant. The main results of the grant were (1) finishing the manuscript of a proof of the completeness of the Poincaré modes (Poincaré, 1885; Bryan, 1889; Lebovitz, 1989; Greenspan, 1990) in an incompressible nonviscous fluid corotating with a rigid ellipsoidal boundary (Backus, 1994a), (2) partial completion of a manuscript describing a definition of helicity that resolved questions in the literature about calculating the helicities of vector fields with complicated topologies (Woltjer, 1958; Moffatt, 1978; Berger and Field, 1984; Backus, 1995a) and (3) the beginning of a reexamination of the inverse problem of inferring properties of the geomagnetic field **B** just outside the CMB from measurements of elements of **B** at and above the earth's surface. This last work has led to a simple general formalism for linear and non-linear inverse problems that appears to include all the inversion schemes so far considered for the uniqueness problem in geomagnetic inversion (Langel, 1991). The technique suggests some new methods for error estimation that form part of this report.

Projects (1) and (2) were described in the Progress Report for NAGW-2967 dated 12/4/92 and in appendices accompanying that report. Work continues on project (2). We describe project (3) in summary below and in more detail in Appendix A of this report. Appendix A is a manuscript accepted by JGR-red on condition that it be shortened and certain mathematical items be omitted. The author felt this would impair the intelligibility of the paper, so he withdrew it and plans to submit a longer version to Geophysical Journal International.

2. THE INVERSION FORMALISM

An inverse problem arises when we measure finitely many data $y^1, \cdots, y^D$ and try to predict from them finitely many numerical properties $z^1, \cdots, z^P$ of the earth. The earth is regarded as an unknown member $\mathbf{x}_E$ of an infinite-dimensional model space X. Since D and P must be finite, we can introduce the data space Y and the prediction space Z consisting respectively of all real D-tuples $\mathbf{y} = (y^1, \cdots, y^D)$ and all real P-tuples $\mathbf{z} = (z^1, \cdots, z^P)$. Both Y and Z are real linear spaces but X may be only a topological space. We assume that we know continuous functions $F: X \to Y$ amd $G: X \to Z$ that would enable us to compute the true data vector $\mathbf{y}_E = F(\mathbf{x}_E)$ and the true prediction vector $\mathbf{z}_E = G(\mathbf{x}_E)$ if we knew the true earth $\mathbf{x}_E$. In fact, we do not know $\mathbf{x}_E$ or even $\mathbf{y}_E$ but only the measured data vector $\mathbf{y}_0 = (y^1, \cdots, y^D)$. It is related to $\mathbf{y}_E$ by

$$\mathbf{y}_0 = \mathbf{y}_E + \delta_R \mathbf{y} + \delta_S \mathbf{y} \tag{1}$$

where $\delta_R \mathbf{y}$ is the random error in the data vector and $\delta_S \mathbf{y}$ is the systematic error. There is no information about these errors except that $\delta_S \mathbf{y} \in V_S$, a known subset of Y, and that $\delta_R \mathbf{y}$ is a realization of a vector random

variable in Y whose probability measure η_R is known except for a few parameters to be determined from the data (Backus, 1989). This inverse problem as stated is well-known to be insoluble for most F and G (Backus and Gilbert, 1967). The remedy is to invoke prior information about $\mathbf{x}_E$ in the form of either a "hard" or a "soft" bound (terminology from Jackson, 1979). These are analogous to systematic and random errors. A hard bound is a subset U_E of X for which we know that $\mathbf{x}_E \in U_E$. A soft bound is a probability measure μ_E on X such that for any measurable subset U of X we assign probability $\mu_E(U)$ to the event $\mathbf{x}_E \in U$. In the geomagnetic problem, an example of a hard bound is that the ohmic heating rate of the core, a quadratic form in $\mathbf{B}$, must be less that the heat flow out of the earth's surface (Parker, 1972; Gubbins, 1983; see, however, Backus, 1975). A bold extrapolator could extract a soft bound, a probability measure on X, from Constable and Parker's (1988) suggestion that the gauss coefficients, the spherical harmonic expansion coefficients of the magnetic scalar potential, are independent normal random variables whose variance depends only on degree l. Constable and Parker applied the Kolmogorov-Smirnov test (Kendall and Stuart, 1979) to degrees $2 \le l \le 12$ because the axial dipole is well-known to be anomalously large, while above $l = 12$ the gauss coefficients of the whole earth are believed to include an appreciable crustal contribution and so do not directly describe the core. Another soft bound might be a Bayesian observer's description of prior beliefs by means of a subjective prior personal probability measure for $\mathbf{x}_E$ in X. Franklin (1970) developed an inversion theory based on soft prior information, but did not discuss the source of that information. Backus (1970a) considered both hard and soft prior bounds and suggested that a hard bound might be replaced by a suitably chosen soft one. Jackson (1979) investigated this "bound-softening" proposal in some detail, and Gubbins (1983) used it to estimate gauss coefficents and their errors at the CMB. It is now known that Backus's suggestion of 1970 was wrong, and that softening a hard quadratic bound necessarily introduces spurious prior "information" about $\mathbf{x}_E$ when $dim\, X = \infty$ (Backus, 1988, 1989a).

Rigourous inversion schemes have been addressed to one of two audiences. The Bayesians are willing to use probabilities to quantify personal beliefs. The frequentists hold that probabilities are meaningless unless they describe frequencies of outcome in experiments that are at least conceptually repeatable. In inversions frequentists harden all soft bounds and random errors by computing confidence sets and confidence intervals (Backus, 1989; Donoho, 1989; Stark, 1992). Hardening soft bounds does not introduce spurious new "information" about $\mathbf{x}_E$ or $\delta_R \mathbf{y}$ (Backus, 1989). Bayesians have tried to soften all hard bounds and systematic errors into probability measures (Jackson, 1979; Gubbins, 1983; Tarantola, 1987; Backus, 1988a). Although this bound softening is now known to be unacceptable in spaces of infinite dimension, one might wonder whether it would work in spaces of finite dimension, like Z. Backus (1995b) shows that in finite dimensional spaces replacing an upper bound on a quadratic form by a normal probability measure introduces a spurious lower bound on the form. That is, the softening generates a claim that one can estimate the actual value of the quadratic form to within a certain factor at a certain error rate. For example, in three dimensions this bound-softening leads to a lower bound that is 1/20 of the upper bound at the 90% confidence level, and 1/3 of the upper bound at the 50% confidence level.

These complications motivate the search for an inversion formalism able to combine hard and soft bounds on $\mathbf{x}_E$ with random and systematic data errors in a way acceptable to both frequentists and Bayesians. The scheme should not require a commitment to either philosophy of probability, and should produce a result that could be interpreted in the light of either. Here we describe what we believe to be such a scheme. In the next sections we describe how we propose to use it in the geomagnetic inversion problem.

Our formalism requires the contstruction of an object that we will call a truncated approximation (TA) to the inverse problem (X, F, G). A TA is a quintuple $(X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$ in which X_{TA} is a subset of a finite-dimensional topological space and the last four entries are functions. Their domains and codomains are as follows.

$$P_{TA}: X \to X_{TA}, \qquad F_{TA}: X_{TA} \to Y, \qquad Q_{TA}: Y \to Y_{TA}, \qquad G_{TA}: X_{TA} \to Z, \tag{2}$$

where $Y_{TA} = F_{TA}(X_{TA})$. We put no restrictions on these functions except to demand that P_{TA} and Q_{TA} be continuous, that the restriction of Q_{TA} to Y_{TA} be the identity map on Y_{TA} and that F_{TA} have a continuous inverse, $F_{TA}^{-1}: Y_{TA} \to X_{TA}$. We define $H_{TA} = G_{TA} \circ F_{TA}^{-1} \circ Q_{TA}$ and

$$f_{TA} = F_{TA} \circ P_{TA} - F\,, \quad g_{TA} = G_{TA} \circ P_{TA} - G\,, \quad \delta_{TA}\mathbf{y} = f_{TA}(\mathbf{x}_E)\,, \quad \delta_{TA}\mathbf{z} = g_{TA}(\mathbf{x}_E)\,. \tag{3}$$

The true value of the desired prediction is $\mathbf{z}_E = G(\mathbf{x}_E)$. Our formalism leads to the conclusion that $\mathbf{z}_E = \mathbf{z}_0 - \delta\mathbf{z}$ where $\mathbf{z}_0 = H_{TA}(\mathbf{y}_0)$ and

$$\delta\mathbf{z} = H_{TA}(\mathbf{y}_0) - H_{TA}(\mathbf{y}_0 - \delta_R\mathbf{y} - \delta_S\mathbf{y} + \delta_{TA}\mathbf{y}) + \delta_{TA}\mathbf{z}\,. \tag{4}$$

Equation (4) simplifies considerably when H_{TA} is linear. It is the final result of the inversion and is offered to frequentists and Bayesians for their own interpretations. It describes the prediction error $\delta\mathbf{z}$ as a combination of systematic and random errors. For example, if the prior information is a hard bound, $\mathbf{x}_E \in U_E$, then $\delta_{TA}\mathbf{z} \in g_{TA}(U_E)$, so $\delta_{TA}\mathbf{z}$ is a systematic error. If the prior information is a probability measure μ_E on X then $\delta_{TA}\mathbf{z}$ is a random error with probability measure $\mu_E \circ g_{TA}^{-1}$. Here g_{TA}^{-1} denotes the mapping from measurable subsets of Z to measurable subsets of X; the point mapping $g_{TA}^{-1}: Z \to X$ does not exist because $dim\ Z < \infty$ and $dim\ X = \infty$. Frequentists can convert the random errors in $\delta\mathbf{z}$ to confidence sets (hard bounds) and thus obtain a confidence set for $\delta\mathbf{z}$ (Backus, 1989; Donoho, 1990; Stark, 1992). Bayesians can convert systematic errors in $\delta\mathbf{z}$ to random errors if $dim\ Z \le 2$. If $dim\ Z \ge 3$, softening hard bounds introduces enough spurious "information" that Bayesians will probably want to describe the error $\delta\mathbf{z}$ as the sum of a systematic error with known confinement set and a random error with known probability measure.

Clearly, to apply the foregoing formalism to any particular problem requires effective computational use of the mathematical and physical structure of that problem. One useful tool in this endeavor is the notion of data compression. Suppose we have M linearly independent linear functionals on Y, $\tilde{K}^i: Y \to R$ for $i = 1, \cdots, M$. (R denotes the field of real numbers.) Let $\tilde{Y}$ be the space of real M-tuples and define the linear function $\tilde{K}: Y \to \tilde{Y}$ by $\tilde{K}(\mathbf{y}) = (\tilde{K}^1(\mathbf{y}), \cdots, \tilde{K}^M(\mathbf{y}))$. If $(X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$ is a truncated approximation for the inverse problem (X, F, G) then $(X_{TA}, P_{TA}, \tilde{K} \circ F_{TA}, \tilde{K} \circ Q_{TA}, G_{TA})$ is a truncated approximation for $(X, \tilde{K} \circ F, G)$. In the geomagnetic problem we will see later that the compression operator $\tilde{K}$ can be chosen so that there are obvious bases in X_{TA} and $\tilde{K}(Y_{TA})$ relative to which the matrix of the compressed data function $\tilde{K} \circ F_{TA}$ is nearly diagonal. This fact makes it easy to trace error propagation and speeds up the matrix inversions required to treat the real data.

Another essential tool in applying the formalism to estimate the error $\delta\mathbf{z}$ in the prediction vector is the availability on the data space Y of a dot product generated by the probability measure η_R for the random error $\delta_R\mathbf{y}$ (Backus, 1989). Relative to this dot product, the variance tensor of $\delta_R\mathbf{y}$ is the identity tensor on Y. This situation is preserved under data compression. If the inverse problem (X, F, G) is replaced by a compressed version $(X, \tilde{K} \circ F, G)$ then the compressed data space $\tilde{Y}$ has a natural dot product generated by the probability measure $\tilde{\eta}_R = \eta_R \circ \tilde{K}^{-1}$ for the compressed random error $\delta_R\tilde{\mathbf{y}} = \tilde{K}(\delta_R\mathbf{y})$. The compression mapping $\tilde{K}$ is an isometry between $\tilde{Y}$ and the orthogonal complement of the null space of $\tilde{K}$.

On the infinite dimensional model space X, a natural dot product can be constructed from the prior information if X is a linear space and the prior information is a hard quadratic bound. Hard non-quadratic prior bounds may make X a Banach space or simply a topological linear space. If the prior bound on $\mathbf{x}_E$ is soft, a probability density μ_E on X, then it generates no natural dot product on X (Freedman, 1963; Backus, 1987). However, in any truncated approximation $(X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$ to (X, F, G), the probability measure $\mu_E \circ P_{TA}^{-1}$ generates a natural dot product on the finite dimensional space X_{TA}. In the inversion, this dot product can be used in place of a dot product on X.

3. THE NEED FOR ERROR ESTIMATES IN GEOMAGNETIC MODELING: BACKGROUND

The grant was focussed on the CMB, but the satellite and surface data cannot be analyzed without considering together all five contributions to the magnetic field measurements: the core, the mantle, the crust, external electric currents and measurement errors. The inversion formalism described above seems to meet rather well the needs of this modelling project.

Each of the five field sources can be modelled deterministically or stochastically, although deterministic models of measurement errors (eg *a priori* estimates of hard bounds) seem unlikely to be as useful as stochastic models in which a few parameters are inferred from the fit to the data.

So far, the PI has tried to incorporate neither induced mantle currents nor external currents into the inversion process. There are some deterministic descriptions of both. McLeod (1992, 1994) and Pulkinnin et al (1995) discuss fields of external currents, and Backus (1982) and Benton and Whaler (1983) discuss mantle currents. In the final analysis of the real data, such descriptions will have to be incorporated, or replaced by stochastic substitutes. If the magnetic jerks (Ducruix et al, 1985) turn out to be real, they will provide a very useful upper bound on mantle conductivity and the contribution of mantle currents to the external field. McLeod (1992, 1994) independently estimates deep mantle conductivity by focusing on the transfer functions from external to internal fields of various harmonic degrees.

To date, the PI's inversion work has been restricted to the core, the crust and measurement errors. Recent advances in all three areas are described in the next section. In the remainder of this section we list some of the geophysical questions about the core whose answers depend on reliable modelling and credible error estimates.

Gubbins and Bloxham (1987) believe that their inversion of the data resolves resemblances between magnetic features in the northern and southern hemispheres of the CMB that suggest the inner-core-grazing Taylor columns in Busse's (1983) weak-field dynamo theory. On the other hand, Glatzmeier and Roberts (1995), in a remarkable simulation of the full dynamo equations, find a strong-field dynamo. Whether the patterns observed by Gubbins and Bloxham are required by the data depends crucially on the errors in estimating the radial field B_r on the CMB. The errors quoted by Gubbins and Bloxham do not settle this issue, being simply an estimate of how well their model fits the data, not of how far it is from the real earth (Backus, 1988a; Stark, 1992).

Another question addressed by B_r on the CMB is the on-going controversy about whether, during periods shorter than one or two centuries, the core fluid velocity $\mathbf{v}$ affects $\mathbf{B}$ approximately as if the core were a perfect electrical conductor. When Roberts and Scott (1965) proposed this "frozen flux" approximation, they pointed out that it reduces the dynamo equation at the CMB to $\partial_t B_r + \nabla_S \cdot (B_r \mathbf{v}_S) = 0$, where ∇_S and $\mathbf{v}_S$ are the tangential parts of ∇ and $\mathbf{v}$ (in fact, $\mathbf{v}_S = \mathbf{v}$ at the CMB). This equation of Roberts and Scott has been the basis for almost all subsequent attempts to estimate $\mathbf{v}_S$ at the CMB. These attempts employ satellite and surface magnetic data to estimate B_r and $\partial_t B_r$ on the CMB. Backus (1968) describes some ways to use those data to test the frozen flux hypothesis, but those tests require error estimates for the B_r and $\partial_t B_r$ inferred at the CMB. Bloxham and Gubbins (1986) believe the data require abandoning the frozen flux approximation. Again, however, their error estimates are measures of how well their B_r on the CMB fits the surface data, not how close it is to the real earth. Indeed, Constable et al (1993) have constructed frozen-flux models that do fit the recent data. The data before 1850 are only directions and produce an incompletely understood intrinsic nonuniqueness in determining B_r even at the earth's surface, much less at the CMB (Proctor and Gubbins, 1990).

If the frozen-flux approximation is accepted, the Roberts and Scott equation extracts from B_r and $\partial_t B_r$ on the CMB information about $\mathbf{v}_S$ there. Backus (1968) described this extractable information as follows: there are two scalar fields, f and g, on the CMB such that there $B_r \mathbf{v}_S = \nabla_S f + \hat{\mathbf{r}} \times \nabla_S g$. Exact and complete values for B_r and $\partial_t B_r$ on the CMB determine f up to an irrelevant constant, and g can be chosen arbitrarily except that

$\nabla_S g = \hat{\mathbf{r}} \times \nabla_S f$ on the null-flux curves (where $B_r = 0$). Since 1968 several physical hypotheses have been advanced to obtain more information about g. Voorhies and Backus (1985) suggested that $\mathbf{v}_S$ might be approximately steady over one or two centuries (more precisely, if τ was the time-scale for changes in $\mathbf{v}_S$ and λ_v and λ_B were typical length scales for $\mathbf{v}_S$ and B_r then $(v_S \tau)^{-1} \ll \lambda_v^{-1} + \lambda_B^{-1}$). Bloxham (1990) suggested that the upper core might be so stably stratified that $\nabla_S \cdot \mathbf{v}_S = 0$. LeMouël et al (1985) suggested that the tangential part of the Lorentz force just below the CMB might be much less than the tangential part of the Coriolis force, so that $\nabla_S \cdot (\mu \mathbf{v}_S) = 0$ on the CMB, μ being the cosine of colatitude. All three physical hypotheses remove much of the ambiguity in $\mathbf{v}_S$ and Voorhies removes it all (Voorhies and Backus, 1985; Backus and LeMouël, 1986; Bloxham, 1990). Unfortunately, the data show that $\nabla_S \cdot \mathbf{v}_S$ and $\nabla_S \cdot (\mu \mathbf{v}_S)$ cannot both vanish, so they cannot be used together (Bloxham, 1990). To verify any of these physical hypotheses, one needs error estimates for B_r and $\partial_t B_r$ on the CMB or for B_r at different epochs.

Gubbins and Bloxham (1987) see in calculated maps of $\mathbf{v}_S$ evidence of local upwelling just under the CMB. They suggest that this indicates local high temperatures in the lower mantle. Again, their error estimates are formal rather than substantive.

It is the PI's conviction that believable error estimates for B_r and perhaps $\partial_t B_r$ at the CMB would either resolve many of these controversies or show that they are not resolvable with present data.

4. PROGRESS ON GEOMAGNETIC MODELING TO DATE UNDER THE GRANT

The PI's long-term goal is to use the satellite and observatory data to develop a defensible model of the sources of the geomagnetic field and the errors of measurement. As a first stage in this program, the existing models deserve careful study. Under NAGW-2967 the PI and his student, Andrew Walker, have begun a reexamination of the power spectrum of the external field, as derived by Langel and Estes (1982) and by Cain et al (1989). We hope to learn enough from this study to undertake the second stage: deciding what parts of the model should be stochastic, what parts should be deterministic, and fitting the model to the original magnetic measurements so as to obtain predictions with error estimates about the geomagnetic field. Our interest is in the CMB, but the project must necessarily develop information about the other field sources. We hope the inferences about sources produced in this first stage will guide the second stage. Then those inferences must be rechecked against the results of the second stage.

Most of our work under NAGW-2967 has been on the first stage. We have produced partial results for the core, the crust and the measuring errors. We describe each in turn.

4A. The Core

For the core the issue is to find credible and useful prior information. One hard bound is now well-known: the total ohmic heat production in the core is probably less than the heat flow out of the earth's surface (Parker, 1972; Gubbins, 1983; Backus, 1988a). There is a possibility that this bound could be exceeded (Backus, 1975), so we propose also to study another hard bound we have recently obtained from the virial theorem: the total magnetic field energy must be less than the unsigned gravitational self-energy of the earth. Backus (1989) showed that the surface and satellite data can provide no information about the value of B_r at single sites on the CMB if the prior information is a hard bound on heat flow or energy. Heat flow bounds do permit estimation of the flux of B_r out of regions P on the CMB that have finite area and a smooth boundary curve ∂P, but the energy bounds do not. It is not known whether an energy bound permits estimating the flux through P when $B_r = 0$ on ∂P (ie if ∂P is a null-flux curve).

Neither of the hard bounds just mentioned is tight, and only the dense distribution of satellite data makes them useful (Backus, 1989). The PI has recently found some evidence for a rather speculative soft bound that may have the advantage that observers who accept it can estimate point values of B_r on the CMB, except

perhaps in a region of Lebesgue measure zero (vanishing total area). This is one of the theoretical questions yet to be addressed. At any rate, this new speculative soft bound is much tighter than the two hard bounds, and so deserves attention. Backus (1994b) discussed it at the 1994 fall AGU meeting, and that discussion is detailed in Appendix B. Here we give a summary. In the rest of subsection 4A., **B** will refer to the field of the core, uncontaminated by other sources. For reasons mentioned later, it is generally believed that at spherical harmonic degrees of 12 or less, the field we see at the surface is mostly the core field.

The new bound on **B** is based on a fresh attempt to estimate the depth to the core magnetic sources via the observed decrease of energy with harmonic degree l in the LML spectrum, the spectrum introduced by Lücke (1957), Mauersberger (1956) and Lowes (1966). The attempt is suggested by the observation of Constable and Parker (1988) that the non-dipole gauss coefficients of the present geomagnetic field at the CMB, as far as they can be detected in the MAGSAT data, appear to be distributed like independent normal random variables with zero mean and variance dependent only on degree. Any random field with such statistics is statistically invariant under all rotations about the center of the earth, a property of the geomagnetic non-dipole field recently suggested also by Courtillot et al (1992).

To describe what we have done and propose to do in this line of attack, we accept the following notation: if x is any random variable, $E[x]$ will denote its expected value. R and C are the fieldss of real and complex numbers, and if $z \in C$ then z^* is its complex conjugate. The spherical surface of radius b concentric with the earth is denoted $S(b)$. We will consistently take $S(a)$ to be the CMB and $S(c)$ to be a sphere near the MAGSAT orbits. Cain et al (1989) take c = 6790 km for MAGSAT. If $f : S(b) \to C$, then $\langle f \rangle_{S(b)}$ is the average of f with respect to area on $S(b)$. For $0 \le l \le \infty$ and $-l \le m \le l$ let Y_l^m be a real or complex spherical harmonic polynomial of degree l and longitudinal order m, normalized so that $\langle Y_l^m (Y_{l'}^{m'})^* \rangle_{S(1)} = \delta_{ll'} \delta_{mm'}$. For $r \ge a$ let $\gamma_l^m(r)$ be the coefficients in the expansion

$$B_r(r\hat{\mathbf{r}}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \gamma_l^m(r)\, Y_l^m(\hat{\mathbf{r}}), \tag{5a}$$

where $\hat{\mathbf{r}}$ is an arbitrary unit vector. The Schmidt semi-normalized gauss coefficients $g_l^m(r)$, the coefficients usually given in published reference models, are related to the γ_l^m by the equation

$$(2l+1)^{1/2}\, \gamma_l^m(r) = (l+1)\, g_l^m(r). \tag{5b}$$

The pre-Maxwell equations assure that $r^{l+2}\gamma_l^m(r)$ is independent of r. On $S(r)$ the LML spectrum is

$$R(r,l) = (l+1) \sum_{m=-l}^{l} |g_l^m(r)|^2. \tag{5c}$$

There seems to be no justification for the factor $(l+1)$ in (5c) except habit and the independent demonstrations by Lücke, Mauersberger and Lowes that $R(r,l) = \langle B_l^2 \rangle_{S(r)}$ where $\mathbf{B}_l$ is the part of **B** produced by spherical harmonics of degree l. This looks like a physically significant quantity, but so far no dynamo theory has singled it out. The statistical model we are about to consider suggests that a different factor would be physically more useful in (5c), a conclusion reached also by Jackson (1994a).

Constable and Parker (1988) observed that if $2 \le l \le 12$ and $-l \le m \le l$ then the present-day values of the quantities $(2l+1)(l+1)^{-1/2}\gamma_l^m(a)$ pass the Kolmogorov-Smirnov (K-S) test for independent identically distributed (iid) normal random variables. Andrew Walker found the same result when he multiplied the $\gamma_l^m(a)$ by a few other factors algebraic in l, so the PI decided to look for a model with more structure, one that might justify a particular choice of factor. A physically reasonable place to start seemed to be the non-dipole part of B_r, because the uniqueness assertion in the Neumann problem makes the external **B** a direct expression of B_r just above the CMB, and this B_r is the same as the B_r produced by the core dynamo just below the fluid boundary layers there. The result of Constable and Parker shows that, at least for $2 \le l \le 12$, B_r can be regarded as a random field on $S(a)$ whose statistics are invariant under all rotations about the center of the earth. Their choice

of multiplier for $\gamma_l^m(a)$ amounts, as we shall see, to the choice of a particular covariance function for $B_r(\mathbf{r})$ on $S(a)$. It appeared reasonable to look first at the simplest covariance function, the delta function. That is, we decided to explore the hypothesis that there was a spherical surface $S(w)$ on which B_r was gaussian white noise. This is equivalent to assuming that the $\gamma_l^m(w)$ are iid gaussian with the same variance, which we label σ^2. We left w as a free parameter because the depth to the sources is often estimated by fitting a straight line to the observed $\ln R(c,l)$ defined by (5c) (Langel and Estes, 1982; Cain et al, 1989), and one might hope that fitting the data with our model would produce $w=a$.

Our rather naive model makes a definite prediction about the LML spectrum. Since $\gamma_l^m(c) = (w/c)^{l+2}\gamma_l^m(w)$, then from (5b,c) we have

$$R(c,l) = (2l+1)(l+1)^{-1}\,(w/c)^{2l+4}\sum_{m=-l}^{l}|\gamma_l^m(w)|^2.$$

Our model makes $R(c,l)$ a random variable, and our hypothesis that B_r is white noise on $S(w)$ leads to

$$\ln R(c,l) = \ln[2(w/c)^4\sigma^2] + l\ln[(w/c)^2] + \ln(2l+1) - \ln(l+1) + \zeta_l$$

where $\exp(2\zeta_l)$ is a chi square random variable with $2l+1$ degrees of freedom. The expected value of ζ_l is $E[\zeta_l] = \psi(l+\frac{1}{2})$ and its variance is $V[\zeta_l] = \partial_z\psi(l+\frac{1}{2})$, where ψ is the digamma function, $\psi(z) = \partial_z \ln\Gamma(z)$ (Abramowitz and Stegun, 1964). Thus

$$E[\ln R(c,l)] = \ln A + l\ln\nu + 2(l+2)\ln(a/c) + \ln(2l+1) - \ln(l+1) + E[\zeta_l], \tag{6a}$$

$$V[\ln R(c,l)] = V[\zeta_l] = \partial_z\psi(l+\tfrac{1}{2}) \tag{6b}$$

where $\nu = (w/a)^2$ and $A = 2\nu^2\sigma^2$.

In the range $2 \le l \le 12$, where on $S(c)$ the non-dipole field of the core may be relatively uncontaminated by the crust, our model makes $\ln R(c,l)$ a random variable with variance $V[\zeta_l]$. The obvious course is to estimate the free parameters A and ν from the observed values of $\ln R(c,l)$ by choosing A and ν to minimize the weighted mean square error,

$$\sum_{l=2}^{12} V[\zeta_l]^{-1}\{\ln R(c,l) - E[\ln R(c,l)]\}^2. \tag{7}$$

Cain very kindly made available by ftp the gauss coefficients used in Cain et al (1989), and with these we computed the A and ν which minimize (7). The minimizing values are $\nu = 0.766 \pm 0.031$ and $A = (4\pm2)\times10^9\,\mathrm{nT}^2$. The error estimates are from the statistical model. The model predicts that the residual (7) is a realization of a random variable distributed approximately like chi-square with 11 degrees of freedom. The actual value of (7) with Cain's gauss coefficients is 7.01. Andrew Walker finds that the best-fitting ν and A produce values of $\gamma_l^m(w)/\sigma$ for $2 \le l \le 12$ and $-l \le m \le l$ that do pass, at the 60 per cent confidence level, the K-S test for iid normal random variables with mean zero and variance 1. Since σ was calculated from the sample, the original analytic K-S test was replaced by a Monte Carlo calculation (Lilliefors, 1967; Mason and Bell, 1986). The uncorrected analytic K-S test gave a confidence level of 70 per cent.

With $\nu = 0.766$ we have $a - w = 436 \pm 60$ km. If there really were a sphere $S(w)$ on which B_r is white noise, it would be well below the fluid boundary layers at the CMB, and we could not use the pre-Maxwell equations to "see" it from $S(c)$. Our model must be abandoned. The calculations it generated, however, have another, more plausible interpretation obtained by extending to the CMB a proposal of Jackson (1990, 1994a) for stochastic modelling of the crust. If there were a white noise sphere $S(w)$ with $w \le a$, then on $S(a)$ the field B_r would be random gaussian with mean 0 and variance given by either of

$$E[\gamma_l^m(a)\gamma_{l'}^{m'}(a)^*] = \sigma^2\nu^{l+2}\,\delta_{ll'}\,\delta_{mm'} \tag{8a}$$

$$E[B_r(a\hat{\mathbf{r}})B_r(a\hat{\mathbf{s}})] = \sigma^2\nu^2\sum_{l=2}^{\infty}(2l+1)\nu^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}}) \tag{8b}$$

where P_l is the l-th Legendre polynomial. If $B_r(a\hat{\mathbf{r}})$ is a random non-dipole field on $S(a)$ with mean 0 whose statistics are invariant under rotation and whose covariance function is (8b), this field will produce exactly the same statistics on $S(c)$ as does the non-dipole field whose statistics are white noise on $S(w)$. We interpret our fit of (6a) to the data on $S(c)$ as an indication not that there is an $S(w)$ below the CMB on which B_r is white noise, but rather that B_r behaves on $S(a)$ statistically like a rotationally invariant random non-dipole field with mean 0 and covariance function (8b).

Any covariance function for B_r on $S(a)$ that depends only on $\hat{\mathbf{r}}\cdot\hat{\mathbf{s}}$ specifies the statistics of B_r as a normal random field on each $S(r)$ with $r \geq a$. The Laplace equation assures that the resulting covariance functions for $r \geq a$ constitute a semigroup (Hille and Phillips, 1957) whose generator is (8b) with $\sigma = 1$ and with the lower limit in that sum replaced by $l = 0$. Its relationship to the semigroup's generating function makes (8b) an algebraically convenient choice for the covariance function of B_r on $S(a)$.

As Jackson (1990, 1994a) is careful to point out for the crust, (8b) for the CMB is probably a rather crude representation of the true covariance function. The only properties of that function captured by (8b) are its amplitude and its correlation length. The series can be summed analytically (Backus, 1986, used by Jackson) and describes a narrow positive peak with broad flat negative wings (see Appendix B, Fig. 9). The half height of the peak occurs at a half width of about $1-\nu$ radians. A precise calculation for $\nu = 0.766$ gives a half width of 12 degrees, corresponding to a correlation length of 750 km on the CMB. If this picture survives, we have captured a statistical property of the geodynamo that, in the past, has been interpreted as an anomalously great depth to the sources. There is an alternative to our suggestion. Cain et al (1989) have brought $S(w)$ nearly to the CMB by correcting the core LML spectrum for the crustal contribution and by choosing for their random function a Funk-Hecke transform (Backus, 1986) of B_r. This transform seems to us less natural than B_r itself as a white noise candidate for locating source depth, although it does have tradition in its favor.

Our whole picture needs further examination, some of which we describe below, and which we propose to carry out as the work continues. If the picture does hold up, and if we are willing to extrapolate it to degrees higher than 12, a preliminary study indicates that the prior information it provides will permit the estimation of point values of B_r on $S(a)$, except for a subset of $S(a)$ with zero Lebesgue measure (area).

4B. **The Crust**

Because of a knee in the LML spectrum around harmonic degree $l = 15$, Langel and Estes (1982) and Cain et al (1989) suggested that the magnetic signal outside the earth is dominated by the core for $l < 15$ and by the crust for $l > 15$. One group of workers (Meyer et al, 1983, 1985; Hahn et al, 1984) modelled the crust by subdividing it into geological provinces of different susceptibilities. They could fit the LML spectrum in the range of harmonic degrees $15 \leq l \leq 35$. A second group (Counil et al, 1991) produced a good fit for $15 \leq l \leq 60$ by using only two provinces, continents and oceans. Jackson (1990) pointed out that neither model fits the actual values of the gauss coefficients in those ranges of l.

If a geological model of the crust were found to fit the satellite data in the range of l where those data are dominated by the crust, one could plausibly extrapolate the gauss coefficents of that model down to $l = 1$ and thus isolate the core gauss coefficients in the range where they obscure the crust. In the absence of such a deterministic model of the crust, Shure et al (1985) treated crustal signals on $S(c)$ as systematic errors. Backus (1989) observed that tighter error bounds on predictions could be obtained by treating the crustal "errors" as random. Backus, however, took the crustal signals measured at different points on $S(c)$ to be independent, thus assuming that the correlation length of $\mathbf{B}$ on $S(c)$ was zero. Jackson (1990) pointed out that a random crustal magnetic signal on $S(c)$ is produced by a random magnetization $\mathbf{M}$ in the crust, and that even if the horizontal correlation length of $\mathbf{M}$ in the crust were zero, the geometry of the Laplace equation would produce a correlation length for $\mathbf{B}$ on $S(c)$ at least as large as $c-b$, $S(b)$ being the surface of the earth.

A random error in $\mathbf{B}$ on $S(c)$ with an appreciable correlation length complicates both the task of numerical

inversion and the attempt to construct simplified models whose behavior gives insight into the numerical results of the real data inversions. The problem is that most inversion schemes invoke the inverse of the variance matrix of the random errors. The inversion scheme in the first section of this proposal uses that inverse in order to calculate the natural dot product on the data space. As Jackson (1990) points out, the variance matrix of the random crustal signals at the measuring sites on $S(c)$ will be a full (non-sparse) matrix when the correlation length of $\delta_R \mathbf{B}$ is appreciable. In geomagnetic modeling, numerical inversion of non-sparse matrices is quite feasible on modern computers, but there are calculations in which one might want the singular value decomposition of such a matrix (Backus, 1989). These do begin to be time-consuming.

Jackson (1994a) has found a physically plausible class of simple stochastic models for $\mathbf{M}$ which partly alleviate the problem of a non-sparse error variance matrix. These models do generate correlations between measurements of $\mathbf{B}$ at any two sites on $S(c)$, but at least they do not generate correlations between different $\gamma_l^m(c)$ in (5a). Jackson's idea is to posit a stochastic $\mathbf{M}$ such that for any positions $\mathbf{r} = r\hat{\mathbf{r}}$ and $\mathbf{s} = s\hat{\mathbf{s}}$ in the crust

$$E\,[\mathbf{M}(\mathbf{r})] = \mathbf{0} \qquad \text{and} \tag{9a}$$

$$E\,[\mathbf{M}(\mathbf{r})\mathbf{M}(\mathbf{s})] = \delta(r-s)\,F(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}})\,\mathbf{I} \tag{9b}$$

where $\mathbf{I}$ is the three-dimensional identity tensor. Equation (9b) is an approximation based on the assumption that the vertical correlation length λ of $\mathbf{M}$ is much shorter than h, the thickness of the magnetized crust. Jackson shows that his conclusions are essentially unchanged if $\lambda \gg h$, so that $\delta(r-s)$ is replaced by h^{-1}.

To exploit (9b), Jackson defines coefficients F_l by writing the Legendre expansion of F in the form

$$F(\mu) = \sum_{l=0}^{\infty} (2l+1) F_l P_l(\mu) \tag{9c}$$

where P_l is the Legendre polynomial of degree l. Clearly (9a) implies

$$E\,[\gamma_l^m(c)] = 0. \tag{10a}$$

Jackson shows that if the magnetized part of the crust lies between the spherical surfaces $S(b)$ and $S(b-h)$ then for $l, l' \ll b/h$ equations (9) imply

$$E\,[\gamma_l^m(b)\gamma_{l'}^{m'}(b)^*] = \mu_0^2\, h\, b^{-2}\, l\, F_{l-1}\, \delta_{ll'}\, \delta_{mm'} \tag{10b}$$

where μ_0 is the magnetic permeability of the vacuum.

Jackson proposes a simple preliminary choice for the $F(\mu)$ in (9), namely that

$$F_l = K\,\nu^l \tag{11}$$

where K and ν are positive constants and $\nu < 1$. One advantage of this choice is that the series (9c) can be summed explicitly (Backus, 1986) to give

$$F(\mu) = K\,(1-\nu^2)\,(1+\mu^2-2\mu\nu)^{-3/2}. \tag{12}$$

Jackson's F has two free parameters, K which determines its amplitude and ν which determines the width of its peak. If we write $\mu = \cos\theta$ then $\theta = 0$ at the center of the peak, and at half peak height, very roughly, $\theta = 1-\nu$ radians if ν is close to 1. Thus Jackson's Ansatz (11) describes the two most important features of any covariance function F.

Many features of Jackson's model (9) are more general than first appears. Schur's lemma (Weyl, 1950, p152) shows that if $\mathbf{M}$ is any random magnetization of the crust whose statistics are invariant under all rotations about the center of the earth, then on $S(c)$

$$E\,[\gamma_l^m(c)] = 0 \qquad \text{and} \qquad E\,[\gamma_l^m(b)\gamma_{l'}^{m'}(b)^*] = \phi_l\, \delta_{ll'}\, \delta_{mm'} \tag{13a}$$

for some constants ϕ_l. The PI has shown that if $\mathbf{M}$ has rotationally invariant statistics, then $E[\mathbf{M}(\mathbf{r})\mathbf{M}(\mathbf{s})]$ has four independent components that produce magnetic signals on $S(c)$. If their vertical correlation lengths are all much less than the thickness of the magnetized crust, then one of them is (9b). If all four have roughly the same horizontal correlation function, and if $15 \ll l \ll b/h$, laborious algebra shows that, approximately,

$$\phi_l = K\, l\, F_{l-1} \tag{13b}$$

for a suitably chosen constant K. Therefore, if the statistics of the crust can be treated as rotationally invariant, Jackson's model (9) seems likely to be adequate.

Although Jackson's model leads to a non-sparse error variance matrix for the data, it does diagonalize the variance matrix of the gauss coefficients. We can exploit this by an appropriately chosen compression of the data. Define vector spherical harmonics on $S(1)$ as follows:

$$\mathbf{U}_l^m(\hat{\mathbf{r}}) = [-\nabla r^{-l-1} H_l^m(\hat{\mathbf{r}})]_{r=1}; \qquad \mathbf{V}_l^m(\hat{\mathbf{r}}) = [\nabla H_l^m(\mathbf{r})]_{r=1}; \qquad \mathbf{W}_l^m(\hat{\mathbf{r}}) = [\mathbf{r}\times\nabla H_l^m(\mathbf{r})]_{r=1}. \tag{14}$$

Then $\langle \mathbf{U}_l^m \cdot (\mathbf{V}_{l'}^{m'})^* \rangle_{S(1)} = \langle \mathbf{U}_l^m \cdot (\mathbf{W}_{l'}^{m'})^* \rangle_{S(1)} = \langle \mathbf{V}_l^m \cdot (\mathbf{W}_{l'}^{m'})^* \rangle_{S(1)} = 0$, $\langle \mathbf{U}_l^m \cdot (\mathbf{U}_{l'}^{m'})^* \rangle_{S(1)} = (l+1)\,\delta_{ll'}\,\delta_{mm'}$, $\langle \mathbf{V}_l^m \cdot (\mathbf{V}_{l'}^{m'})^* \rangle_{S(1)} = l\,\delta_{ll'}\,\delta_{mm'}$, $\langle \mathbf{W}_l^m \cdot (\mathbf{W}_{l'}^{m'})^* \rangle_{S(1)} = l(l+1)(2l+1)^{-1}\,\delta_{ll'}\,\delta_{mm'}$. On $S(c)$ the magnetic field is

$$\mathbf{B}(c\hat{\mathbf{r}}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} [g_l^m(c)\,\mathbf{U}_l^m(\hat{\mathbf{r}}) + h_l^m(c)\,\mathbf{V}_l^m(\hat{\mathbf{r}}) + k_l^m(c)\,\mathbf{W}_l^m(\hat{\mathbf{r}})] \tag{15}$$

where the terms in $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ describe respectively the fields generated by currents inside, outside and crossing $S(c)$ (Stern, 1976; Kosik, 1984; Backus, 1986). Backus shows that the $g_l^m(c)$ are the Schmidt semi-normalized gauss coefficients. From the orthogonality relations for the vector spherical harmonics (14)

$$(l+1)\,g_l^m(c) = (4\pi)^{-1} \int_{S(1)} dA(\hat{\mathbf{r}})\, \mathbf{B}(c\hat{\mathbf{r}}) \cdot \mathbf{U}_l^m(\hat{\mathbf{r}})^* . \tag{16}$$

Equation (16) suggests a way to compress the data. Suppose $\mathbf{B}$ has been measured at sites $\mathbf{r}_1, \cdots, \mathbf{r}_{D/3}$, so that the data space Y has dimension D. Write $\mathbf{r}_i = r_i\hat{\mathbf{r}}_i$ and assign to each unit vector $\hat{\mathbf{r}}_i$ a small patch ω_i on $S(1)$ with area $4\pi\,|\,\omega_i\,|$. We ask that $\hat{\mathbf{r}}_i \in \omega_i$ and that every $\hat{\mathbf{r}} \in S(1)$ belong to exactly one ω_i. Choose an integer L for which $L(L+2) < D/3$, and for all l, m such that $1 \le l \le L$ and $-l \le m \le l$ define $\tilde{g}_l^m(c)$ by

$$(l+1)\,\tilde{g}_l^m(c) = \sum_{i=1}^{D/3} |\,\omega_i\,|\,(r_i/c)^{l+2}\,\mathbf{B}(\mathbf{r}_i) \cdot \mathbf{U}_l^m(\hat{\mathbf{r}}_i)^* . \tag{17}$$

Evidently $\tilde{g}_l^m(c)$ depends linearly on the data. Usually, the $\tilde{g}_l^m(c)$ for different l and m will be linearly independent functionals on Y. Taken together, they will produce a data compression $\tilde{K}: Y \to \tilde{Y}$, where $\tilde{Y}$ is the space of $L(L+2)$-tuples of complex numbers (or real numbers if the harmonics H_l^m were chosen to be real). If the observation sites $\mathbf{r}_i$ are all nearly on and more or less evenly distributed over $S(c)$, and if the $\mathbf{B}$ in (15) does not include measurement error, $\tilde{g}_l^m(c)$ is likely to be a reasonable approximation to $g_l^m(c)$. Then the statistical properties of the contribution to $\tilde{g}_l^m(c)$ from a rotationally invariant random crust will be approximately (10). We have begun a study of the accuracy of this approximation, but much remains to be done. Preliminary results suggest that the approximation will be acceptable for $L \le 45$. For higher L it appears likely that the data are dominated by the errors of measurement (Cain et al, 1989).

Langel et al (1982a) have shown how to use satellite data at different epochs to eliminate local crustal anomalies from data taken at magnetic observatories. The availability of satellite vector data at two different epochs (MAGSAT adn OERSTED) makes the proposal of Langel et al particularly useful.

4C. The Errors of Measurement

The simplest assumption about the errors in the D observations of components of $\mathbf{B}$ at the $D/3$ locations $c\hat{\mathbf{r}}_1, \cdots, c\hat{\mathbf{r}}_{D/3}$ is that those errors are iid gaussian with mean 0 and variance σ^2. Then the errors of measurement, $\delta_R\mathbf{B}$, are governed by the statistics

$$E\,[\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_i)\,] = 0 \qquad E\,[\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_i)\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_j)\,] = \sigma^2 \mathbf{I}\,\delta_{ij}\,. \tag{18}$$

If we want to use the data compression (17) in order to deal with the non-zero correlation length of the crustal signal, we must also see how that compression affects the errors of measurement. Let $\delta_R \tilde{g}_l^m(c)$ be the error in $\tilde{g}_l^m(c)$ produced by the measurement errors. Then equation (17) implies

$$(l+1)(l'+1)E\,[\,\delta_R \tilde{g}_l^m(c)\,\delta_R \tilde{g}_{l'}^{m'}(c)\,] = \sigma^2 \sum_{i=0}^{D/3} |\,\omega_i\,|^2\ \mathbf{U}_l^m(\hat{\mathbf{r}}_i)\cdot \mathbf{U}_{l'}^{m'}(\hat{\mathbf{r}}_j)^*\,. \tag{19}$$

If the $|\,\omega_i\,|$ are all nearly the same, they are all nearly $D/3$. Then we can approximate the sum in (19) by an integral over $S(1)$ and obtain

$$(l+1)E\,[\,\delta_R \tilde{g}_l^m(c)\,\delta_R \tilde{g}_{l'}^{m'}(c)\,] = \sigma^2 (D/3)^{-1}\,\delta_{ll'}\,\delta_{mm'}\,. \tag{20}$$

Holme and Bloxham (1995) have made a very important improvement in modelling the statistics of the errors of measurement. They point out that $\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_i)$ consists of three parts: magnetometer errors, satellite tracking errors, and satellite orientation errors. The preceding paragraph describes only the magnetometer errors. Holme and Bloxham note that tracking errors may introduce correlations for $i \neq j$, but they have not yet studied these. For small orientation errors in which all three Euler angles at each site are iid with variance θ^2 radians^2, they have shown that

$$E\,[\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_i)\,] = 0, \qquad E\,[\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_i)\,\delta_R \mathbf{B}(c\,\hat{\mathbf{r}}_j)\,] = \theta^2 (B^2\mathbf{I} - \mathbf{BB})\,\delta_{ij}\,, \tag{21}$$

where $\mathbf{B}$ is the geomagnetic field at $c\,\hat{\mathbf{r}}_i$. The full error-variance tensor for the compressed data $\tilde{g}_l^m(c)$ is the sum of (18) and (21) plus whatever contribution comes from tracking errors. It is easy to show that, if $\mathbf{B}$ is approximated by the axial dipole field of the earth, then the effect of (21) on the errors in the compressed data (17) is to introduce nonzero correlations between $\delta_R \tilde{g}_l^m(c)$ and $\delta_R \tilde{g}_{l+2}^m(c)$ for all l and m.

4D. A First Stage Synthesis

Cain et al (1989) concluded that the anomalously large depth below the CMB previously found for the magnetic sources in the core (174 km, Langel and Estes, 1982; 147 km, Meyer et al, 1983) could be largely removed by including the crustal contribution to the LML spectrum $R(c,l)$ at all spherical harmonic degrees l, even those so low ($l \leq 14$) that the core signal dominated the satellite data. This observation by Cain et al has the added attraction that it suggests we can use extrapolation to estimate the crustal contribution to $R(c,l)$ in the range of l dominated by the core. On the other hand, the statistical model we propose in section 4A calls for an apparent source depth 436 km below the CMB. This is so large that we prefer to interpret it as representing a statistical regularity of B_r on the CMB rather than a white noise source for B_r below the CMB. Our model is supported by the fact that the gauss coefficients with $2 \leq l \leq 12$ do pass the Kolmogorov-Smirnov (K-S) test for randomness posed by the model. However, the model described in section 4A has no crustal correction, and the work of Cain et al (1989) suggests that such a correction may be important. This exploration is part of the work we plan for the future. We have done some preliminary calculations, which we set forth here.

If we omit the correction for satellite orientation developed by Holme and Bloxham (1995), the statistical models in sections 4A, 4B, 4C predict that the LML spectrum $R(c,l)$ is the realization of a random variable whose expected value varies with l like

$$E\,[R(c,l)] = (2l+1)[\,(2l+1)(l+1)^{-1}\,A\,\alpha^l + l\,(l+1)B\,\beta^l + C\,] \tag{22}$$

where A, α, B, β, C are contstants, A and α coming from the core, B and β from the crust and C from errors of measurement. On the other hand, Cain et al (1989) fit the observed $R(c,l)$ with the traditional expression,

$$R(c,l) = A\,\alpha^l + B\,\beta^l + C\,. \tag{23}$$

We have made some preliminary fits of (22) to the observed $R(c,l)$ reported by Cain et al (1989). We considered two ranges of l: $2 \le l \le 40$ and $2 \le l \le 59$, and we used two different fitting techniques: maximum likelihood and weighted least squares with weights inversely proportional to the variances predicted by our model. If $S(w)$ is the apparent "white noise sphere" in the core and $S(a)$ is the CMB, the crustal correction reduces the apparent source depth $a - w$ from 436 km to between 240 km and 320 km, depending on the range of l and the fitting method. These values still seem too large to permit accepting $S(w)$ as physically real. Even after the crustal correction we maintain our view that what we see in the LML spectrum is not a white noise source but rather the statistical regularity of B_r on $S(a)$. The crustal correction reduces the correlation length of B_r on $S(a)$ from 750 km to between 370 km and 500 km, depending on the range of l and the fitting method.

Fitting (22) to the observed satellite power spectrum also gives values for the crustal parameters B and β and the parameter C describing instrument error. For the crustal magnetization, we find a horizontal correlation length which lies between 210 and 540 km, whereas Jackson (1994a) found 50 km. Our results are preliminary, and we are not even sure yet that our program is reliable. However, Jackson used the model of Cain et al (1989) described by (23), which may differ enough from our model (22) to explain the difference in correlation lengths. We should also note (A. Jackson, private communication) that Jackson tested the randomness of Cain et al's (1989) individual $g_l^m(c)$ in $16 \le 45$ by normalizing them with Cain's values of $R(c,l)$. This amounts to determing 30 parameters of the distribution from the data, and may require a Monte Carlo amendment of the direct K-S test of randomness used by Jackson. Perhaps he will repeat his test, and we plan to do so. Our own randomness test fails at the moment. When we divide Cain's gauss coefficients by their standard deviations, as derived from the best fitting (22), the normalized coefficients fail the gaussian K-S test for $2 \le L$ if L is much larger than 20. Preliminary computations indicate that the normalized gauss coefficients with $2 \le l \le 40$ pass the test if we delete those with $15 \le l \le 25$. Jackson (1994a) raised the possibility that the crust may not be well described by a rotationally invariant stochastic model. Barring program errors, we may be seeing indications of what he suggested. There may be crustal randomness above $l = 15$ and geography below. This whole area needs much more work.

It remains to discuss the C obtained in the fit of (22) to the observed satellite LML spectrum. As noted in section 4C, if we omit the treatment of satellite orientation errors developed by Holme and Bloxham (1995), the value of C should be approximately $\sigma^2 (D/3)^{-1}$, where σ^2 is the variance of the error in measuring one component of **B** at one site, and D is the number of such components in the data vector. Cain and Langel (private communications) both suggested that $D \approx 50{,}000$ was a reasonable estimate for MAGSAT. Then our C's gave values of σ between 3 and 7 nT, depending on the range of l and the fitting method. Langel et al (1982b) estimated σ by a quite different procedure: combining engineering estimates of the various sources of error in the instruments and the satellite orientation (which they assumed would produce an isotropic random error in **B**). They found $\sigma = 5.8$ nT.

Our preliminary results suggest a number of lines of investigation in the first stage of the work. We must explore our numerical techniques very carefully, because both of our penalty functions appear to have several minima. Considerable exploration of parameter space will be required before we can have any confidence that a rotationally invariant stochastic model of the crust must be abandoned. If that turns out to be the case, we must look for anisotropic stochastic crustal models. If that search fails, we must determine how much of the crustal signal can be modelled stochastically and how much requires deterministic modelling. A statistical model of the crust or a deterministic geological model would lend themselves to extrapolation to the low l where the core is visible magnetically. It is hard to see how, by themselves, deterministic gauss coefficients in the range $l \ge 15$ could do so, even if we were confident that they represented the crust.

It would be very interesting to see whether we can detect in the data the difference between (22) and its modification when corrected for Holme and Bloxham's (1995) treatment of satellite orientation error. Such

detection seems unlikely with existing gauss coefficients, since their authors mistrust them above $l \approx 40$ (Cain and Langel, private communications), while the crustal signal appears to dominate the measurement errors below $l \approx 50$ (Arkani-Hamed and Strangway, 1986; Cain et al, 1989).

Perhaps the most interesting consequence of a convincing statistical model for the "visible" gauss coefficients of the core ($2 \le l \le 12$) would be that the boldest observers might be willing to extrapolate this core model to higher l, where the core is concealed by the crustal signal or measurement noise. Such a statistical core model would provide a soft prior bound on the deterministic core model $\mathbf{x}_E$ that was much sharper than either of the hard prior bounds, ohmic heat production or magnetic energy. As already mentioned, this soft prior bound may be strong enough to permit estimation of B_r at most points on the CMB, a possibility foreclosed by both hard prior bounds.

5. FUTURE PLANS

The first stage of this project involves working with the gauss coefficients already available in the literature, in order to gain some understanding of whether the soft prior bound on $\mathbf{x}_E$ is defensible, whether the crustal signal can be treated as partly or fully stochastic, and what is the structure of the errors of measurement. In the second stage of our work, we hope to generate our own gauss coefficients from the MAGSAT and OERSTED meaurements of **B** in near-earth orbit. (The PI has been accepted as an OERSTED investiagator.) An essential part of this process will be to use the inversion scheme described in Section 2 to estimate the systematic and random errors in our gauss coefficients. In our truncated approximation to the geomagnetic inverse problem, the truncated model space X_{TA} will usually be the space of gauss coefficients up to some maximum degree, although Constable et al (1993) have found it useful in calculating flux on the CMB to represent the core field there by a continuous, piecewise-linear approximation to its radial component. Our inversion formalism applies to any method for approximating the field sources, and we plan to try several.

In addition to the satellite data, our second stage inversion will use what we have learned in the first stage about possible models for the core field, the crustal field and the errors in measuring **B**. The second stage will also require modelling fields from the ionosphere and magnetosphere, since they do contribute measurably to **B** (Langel and Estes, 1985; Cain et al, 1989; McLeod, 1992, 1994).

MAGSAT measured all three components of **B**, and that is also the plan for OERSTED, so the part of our inversion which produces the gauss coefficients can be linear. The estimated errors in the MAGSAT component data were three times those in the intensity data (Langel et al, 1982), so we will try to construct a truncated approximation to the inverse problem which uses the intensity data as well as the components. We plan first to try iterative schemes based on linearization, as is usually done in non-linear least squares. Component data must be included when using the intensities because it is known that with intensities alone there is no useful truncated approximate of the inverse problem (Backus, 1970b).

In estimating core gauss coefficients in the second stage we propose to reexamine the literature on magnetic jerks and long-period magnetic induction (Langel and Estes, 1985; Ducruix et al, 1980; McLeod, 1992, 1994) in the hope of estimating an upper bound on the electrical conductivity in the lower mantle. Previous work (Benton and Whaler, 1983) suggests that mantle conductivity may not much affect main field models, and that its main effect is simply to delay core signals by perhaps a decade or less in their upward passage from the CMB to the surface (Backus, 1982). It is conceivable that the two satellites will give us error bounds on the core gauss coefficients tight enough to justify including mantle currents in the inversion.

Among the questions we hope to answer in the second stage are these: what is the highest degree l at which the satellite data give us whole earth gauss coefficients larger than their error estimates? Do these gauss coefficients support the conclusions reached in the first stage of the work? For example, does our statistical

model of B_r at the CMB remain viable, perhaps with some changes in parameters? Can part of the crustal field be treated stochastically, and, if so, is it a part which admits plausible extrapolation of its statistics to the range of harmonic degrees between 1 and 14, where the crustal signal is obscured by the core. If so, then we can at least estimate how much random error the crust contributes to the core gauss coefficients, even if we cannot separate the core and crustal parts of those coefficients. If the crust cannot be treated stochastically in the range of degrees where we see the core coefficients ($1 \le l \le 14$ or less), we can try to get hard bounds on the errors in the core coefficients from hard bounds on crustal magnetization, or we can join the attempts to find a deterministic geological model. At the moment we have no good ideas for this latter project, but other workers are pursuing it actively (Whaler, 1994).

Since the MAGSAT and OERSTED satellite data will be separated by at least 15 years, we hope that our error bounds at the CMB will be tight enough to throw light on some of the outstanding theoretical controversies which involve predictions about time behavior. Can null-flux curves be seen to appear or disappear, or can the flux through them be estimated accurately enough, to rule out the frozen-flux hypothesis of Roberts and Scott (1965)? If not, can the areas inside null-flux curves be estimated accurately enough to test their constancy in time, as required by Bloxham's (1990) suggestion that the core fluid velocity just below the boundary layer at the CMB might be toroidal? Can the areas inside the projections of null-flux curves onto the equatorial plane be estimated accurately enough to test their constancy in time, as required (Jackson, 1994b) by the proposal of Le Mouël et al (1985) that the fluid velocity just below the CMB boundary layer is radially geostrophic?

BIBLIOGRAPHY

Abramowitz, M. and Stegun, I.A., 1964. *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series **55**, U.S. Government Printing Office, 1046 pp.

Arkani-Hamed, J. and Strangway, D.W., 1986. Band-limited global scalar magnetic anomaly map of the earth deduced from Magsat data, *J. Geophys. Res.*, **91**, 8193–8203.

Backus, G.E., 1968. Kinematics of secular variation in a perfectly conducting core, *Phil. Trans. R. Soc. London A*, **263**, 239–266.

Backus, G.E., 1970a. Inference from inadequate and inaccurate data, I, *Proc. Nat. Acad. Sci.*, **65**, 1-7, 1970a.

Backus, G.E., 1970b. Non-uniqueness of the external geomagnetic field determined by surface intensity measurements, *J. Geophys. Res.*, **75**, 6339–6341.

Backus, G.E., 1975. Gross thermodynamics of heat engines in deep interior of earth, *Proc. Nat. Acad. Sci. USA* **72**, 1555–1558.

Backus, G.E., 1982. The electric field produced in the mantle by the dynamo in the core, *Phys. Earth Planet. Int.*, **28**, 191–214.

Backus, G.E., 1986. Poloidal and toroidal fields in geomagnetic field modeling, *Rev. Geophys.*, **24**, 75–109.

Backus, G.E., 1987. Isotropic probability measures in infinite dimensional spaces, *Proc. Natl. Acad. Sci. USA*, **84**, 8755–8757.

Backus, G.E., 1988a. Bayesian inference in geomagnetism, *Geophys. J.*, **92**, 125–142.

Backus, G.E., 1988b. Comparing hard and soft prior bounds in geophysical inverse problems, *Geophys. J.*, **94**, 249–261.

Backus, G.E., 1989. Confidence set inference with a prior quadratic bound, *Geophys. J.*, **97**, 119-150.

Backus, G.E, 1994a. Completeness of the Poincaré modes of an incompressible non-viscous fluid in a corotating ellipsoid, *Mathematika*, submitted.

Backus, G.E., 1994b. An estimate of magnetic correlation length at the core-mantle boundary from the Lücke-Mauersberger-Lowes spectrum, *EOS*, **75**, 84.

Backus, G.E., 1995a. Evaluating helicity from an unambiguous defifition, in preparation.

Backus, G.E., 1995b. Truncation, Approximation and Procrastination as Inversion Techniques, in preparation.

Backus, G.E. and F. Gilbert, 1967. Numerical applications of a formalism for geophysical inverse problems, *Geophys. J.R. astr. Soc.*, **13**, 247–276.

Benton. E.R. and Whaler, K.A., 1983. Rapid diffusion of the poloidal geomagnetic field through the weakly conducting mantle: a perturbation solution, *Geophys. J.*, **75**, 77–100.

Berger, M.A. and Field, G.B., 1984. The topological properties of magnetic helicity, *J. Fluid Mech.*, **147**, 133–148.

Bloxham. J., 1990. On the consequences of strong stable stratification at the top of earth's outer core, *Geophys. Res. Letters*, **12**, 2081–2084.

Bloxham, J., and D. Gubbins, 1986. Geomagnetic field analysis—IV. Testing the frozen flux hypothesis, *Geophys. J. R. Astron. Soc.*,, **84**, 139–152.

Bloxham, J., Gubbins, D. & Jackson, A., 1989. Geomagnetic secular variation, *Phil. Trans. Roy. Soc. London A*, **329**, 415–502.

Bloxham, D. and Jackson, A., 1991. Fluid flow at the earth's core surface, *Rev. Geophys.*, **29**, 97–120.

Bryan, G.H., 1889. The waves on a rotating liquid spheroid, *Phil. Trans. Roy. Soc. London A*, **180**, 187–219.

Busse, F., 1983. Recent developments in the dynamo theory of planetary magnetism, *Ann. Rev. Eath Planet. Sci.*, **11**, 241–268.

Cain, J.C., Z. Wang, C. Kluth and D.R. Schmitz, 1989. Derivation of a geomagnetic model to n=63, *Geophys. J.*, **97**, 431–442.

Constable, C.G. and Parker, R.L., 1988. Statistics of the geomagnetic secular variation for the past 5 m.y. *J. Geophys. R*, **93**, 11569–11581.

Constable, C.G., Parker, R.L. and Stark, P.B., 1993. Geomagnetic fields incorporating frozen flux constraints. *Geophys. J. Int.*, **113**, 419–433.

Counil, J.L., Achache, J. & Cohen, Y., 1991. The global continent-ocean magnetization contrast, *Earth Planet. Sci. Letters*, **103**, 354–364.

Courtillot, V., Valet, J.-P., Hulot, G. and LeMouël, J.-L., 1992. The earth's magnetic field: which geometry? *EOS*, **73**, 337–342.

Donoho, D.L., 1989. Statistical estimation and optimal recovery, *Technical Report No. 249, Department of Statistics*, University of California, Berkeley.

Ducruix, J., Courtillot, V. & LeMouël, J.-L., 1980. The late 1960s secular variation impulse, the eleven year magnetic variation adn the electrical conductivity of the deep mantle, *Geophys. J. R. astr. Soc.*, **61**, 73–94.

Franklin, J.N., 1970. Well-posed stochastic extension of ill-posed problems. *J. Math. Anal. Appl.*, **31**, 682–716.

Freedman, D.A. 1963. Invariants under mixing which generalize De Finetti's theorem: continuous time parameter, *Annals of Math. Stat.*, **34**, 1194– 1216.

Glatzmeier, G., and Roberts, P.H., 1995. A three-dimensional convective dynamo solution with rotating and finitely conducting inner core and mantle, *Phys. Earth. Plan. Int.*, in press.

Greenspan, H.P., 1990. *The Theory of Rotating Fluids*, Breukelen Press, Brookline, MA, 325 pp.

Gubbins, D., 1983. Geomagnetic field analysis - I. Stochastic inversion, *Geophys. J.R. astr. Soc.*, **73**, 641–652.

Gubbins, D. and Bloxham, J., 1987. Morphology of the geomagnetic field and implications for the geodynamo, *Nature*, **325**, 509–511.

Hahn, A., Ahrendt, H., Meyer, J. & Hufen, J.H., 1984. A model of magnetic sources within the earth's crust compatible with the field measured by the satellite Magsat, *Geol. Jb. Hanover*, **75**, 125–156.

Hide, R. and Roberts, P.H., 1979. How strong is the magnetic field in the earth's liquid core? *Phys. Earth*

Planet. Int., **20**, 124–126.

Hille, E. and Phillips, R.S., 1957. *Functional Analysis and Semi-Groups*, Am. Math. Soc. Colloq. Pub. 31, Providence, RI, 808 pp.

Holme, R. and Bloxham, J., 1995. Alleviation of the Backus effect in geomagnetic field modelling *Geophys. Res. Letters*, in press.

Jackson, A., 1990. Accounting for crustal magnetization in models of the core magnetic field, *Geophys. J. Int.*, **103**, 657–674.

Jackson, A., 1994a. Statistical treatment of crustal magnetization, *Geophys. J. Int., In press.*

Jackson, A., 1994b. The magnetic field as a probe of the earth's core. *EOS*, **75**, 84.

Jackson, D., 1979. The use a priori data to resolve non-uniqueness in linear inversion, *Geophys. J. R. Astron. Soc.*, **57**, 137–158.

Kendall, M. and Stuart, A., 1979. *The Advanced Theory of Statistics, Volume 2* MacMillan, New York, 748 pp.

Kosik, J.C., 1984. Quantitative magnetospheric magnetic field modeling with toroidal and poloidal vector fields, *Planet. Space Sci.*, **31**, 965–974.

Langel, R., 1991. Recent progress in estimating uncertainty in geomagnetic field modeling, *Geophys. Astrophys. Fluid Dynamics*, **60**, 37–88.

Langel, R. and Estes R., 1982. A geomagnetic field spectrum, *Geophysical Research Letters*, **9**, 250–253.

Langel, R. and Estes R., 1985. Large-scale, near-earth magnetic fields from external sources and the corresponding induced internal field, *J. Geophys. Res.*, **90**, 2487–2494.

Langel, R., Estes R. and Mead, G.D., 1982a. Some new methods in geomagnetic field modeling applied to the 1960–1980 epoch, *J. Geomag. Geoelectr.*, **34**, 327–349.

Langel, R., Ousley, G. and Berbert, J., 1982b. The Magsat Mission, *Geophysical Research Letters*, **9**, 243–245.

Lebovitz, N.R., 1989. The stability equations for rotating inviscid fluids: Galerkin methods and orthogonal bases, *Geophys. Astrophys. Fluid Dynamics*, **46**, 221–243.

LeMouël, J.-L., Gire, C. & Madden, T., 1985. Motions of the core surface in the geostrophic approximation, *Phys. Earth Planet. Int.*, **39**, 270-287.

Lilliefors, H.W., 1967. On the Kolmogorov-Smirnov test for normality with mean and variance unknown, *J. Am. Statistical Assoc.*, **62**, 399–402.

Lowes, F.J., 1966. Mean-square values on sphere of spherical harmonic vector fields, *J. Geophys. Res.*, **71**, 279.

Lücke, O., 1957. Ueber Mittelwerte von Energiedichtender Kraftfelder, *Wiss. Z. Päd. Hochschule Potsdam, Math.-Nat. Reihe* **3**, 39–46.

Mason, A.L., and Bell, C.B., 1986. New Lilliefors and Srinivasan tables with applications, *Commun. Statist.-Simul.*, **15**, 451–477.

Mauersberger, P., 1986. Das Mittel der Energiedichte des geomagnetischen Hauptfeldes an der Erdoberfläche und seine säkulare Aenderung, *Gerlands Beitr. Geophys.*, **65**, 207–215.

McLeod, M.G., 1992. Signals and noise in magnetic observatory annual means: mantle conductivity and jerks, *J. Geophys. Res.*, **97**, 17261–17290.

McLeod, M.G., 1994. Magnetospheric and ionospheric signals in magnetic observatory monthly means: Electrical conductivity of the deep mantle, *J. Geophys. Res.*, **99**, 13577–13590.

Meyer, J., Hufen, J.H., Siebert, M. & Hahn, A., 1983. Investigations of the internal geomagnetic field by means of a global model of the earth's crust, *J. Geophys.*, **52**, 71–84.

Meyer, J., Hufen, J.H., Siebert, M. & Hahn, A., 1985. On the identification of Magsat anomaly charts as cruastal part of the internal field, *J. Geophys. Res.*, **90**, 2537–2541.

Moffat, H.K., 1978. *Magnetic Field Generation in Electrically Conducting Fluids* Cambridge University Press, Cambridge, 342 pp.

Parker, R.L., 1972. Inverse theory with grossly inadequate data, *Geophys. J. R. Astron. Soc.*, **29**, 123–138.

Poincaré, H., 1885. Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation, *Acta Mathematica*, **7**, 259–380.

Proctor, M.R.E. and Gubbins, D., 1990. Analysis of geomagnetic directional data, *Geophys. J. Int.*, **100**, 69–78.

Pulkinnen, T.I., Baker, D.N., Walker, R.J., Raeder, J. & Ashour-Abdalla, M., 1995. Comparison of empirical magnetic field models and global MHD simulations: the near-tail currents, *Geophys. Res. Letters*, **22**, 675–678.

Roberts, P.H. and Scott, S., 1965. *J. Geomag. Geoelect.*, **17**, 137–151.

Shure, L., Parker, R. & Langel, R., 1985. A preliminary harmonic spline model from magsat data, *J. Geophys. Res.*, **90**, 11505–11512.

Stark, P.B., 1992. Minimax confidence intervals in geomagnetism, *Geophys. J. Int.*, **108**, 329–338.

Stern, D.P., 1976. Representation of magnetic fields in space, *Rev. Geophys.*, **14**, 199–214.

Tarantola, A., 1987. *Inverse Problem Theory*, Elsevier, New York, 613 pp.

Voorhies, C. and Backus, G.E., 1985. Steady flows at the top of the core from geomagnetic field models: the steady motion theorem, *Geophys. Astrophys. Fluid Dynamics*, **32**, 163–173.

Whaler, K., 1994. Direct inversion for crustal magnetization from satellite magnetic field measurements, *EOS*, **75**, 197.

Weyl, H. 1950. *The Theory of Groups and Quantum Mechanics*, Dover, New York, 422 pp.

Woltjer, L., 1958. A theorem on force-free magnetic fields, *Proc. Nat. Acad. Sci.*, **44**, 489–491.

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APPENDIX A

Truncation, Approximation and Procrastination as Inversion Techniques

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By examining the processes of truncating and approximating the model space, and by committing to neither the objectivist nor the subjectivist interpretation of probability, we construct a formal scheme for solving linear and nonlinear geophysical inverse problems. The necessary prior information about the correct model $\mathbf{x}_E$ can be either a collection of inequalities or a probability measure describing where $\mathbf{x}_E$ was likely to be in the model space X before the data vector $\mathbf{y}_0$ was measured. The results of the inversion are i) a vector $\mathbf{z}_0$ that estimates some numerical properties $\mathbf{z}_E$ of $\mathbf{x}_E$; ii) an estimate of the error $\delta\mathbf{z} = \mathbf{z}_0 - \mathbf{z}_E$. Since $\mathbf{y}_0$ is finite dimensional, so is $\mathbf{z}_0$, and hence in principle inversion cannot describe all of $\mathbf{x}_E$. The error $\delta\mathbf{z}$ is studied under successively more specialized assumptions about the inverse problem, culminating in a complete analysis of the linear inverse problem with a prior quadratic bound on $\mathbf{x}_E$. As an idealized example we study the magnetic field at the core-mantle boundary, using satellite measurements of field elements at sites assumed to be almost uniformly distributed on a single spherical surface.

1. INTRODUCTION

As part of a project for inverting data from the magnetic satellites MAGSAT and OERSTED, the authors have developed a general formalism for estimating errors in geophysical inverse problems. This formalism applies to non-linear as well as linear problems, to regularization (Tikhonov, 1963), to hard and soft prior information, and to both Bayesian and frequentist treatments of that prior information. The formalism appears to include and unify many of the techniques used in the last 25 years of geomagnetic inversion (Franklin, 1970; Backus, 1970a,b,c, 1988a,b, 1989; Jackson, 1979; Langel, 1982; Gubbins, 1983, 1984, 1985; Tarantola, 1987; Cain et al, 1989; Stark, 1992; Donaho, 1992). In the present paper we describe and illustrate this formalism in the hope that it will be of interest to others working with data inversion.

The formalism shows how to find error bounds, but does not by itself assure that they will be small enough to be interesting. Therefore we emphasize at the outset that we have not yet applied the formalism to any non-linear problems. Non-linear inversions are usually problem-specific and often involve deep issues of"hard" analysis. We have examined none of these issues. Our reason for describing the formalism in enough generality to cover non-linear problems is that, as often happens, generality simplifies.

Our illustration of the formalism will be an idealized linear MAGSAT inversion rather like that in Backus (1989). The idealized measurement sites are assumed to be almost uniformly distributed on a single spherical surface about 420 km above the earth's surface. As even this idealized linear example will illustrate, the investigator has a wide choice of ways to implement the formalism. There are, however, some aspects of the formalism that can be developed in considerable detail for general linear problems.

Since the formalism aims to be useful to both Bayesians and frequentists, we begin by describing how we use those terms. Frequentists (objectivists) hold with Neymann (1937) that probability has no experimental meaning except as an estimate of frequencies of various outcomes in a repeatable series of random trials. Bayesians (subjectivists) hold that probability distributions can serve as quantitative descriptions of their personal beliefs. Objectivists have sought to construct multidimensional confidence sets for a collection of numerical predictions about the correct earth model $\mathbf{x}_E$ (Backus, 1989; Stark, 1992). Subjectivists have imposed prior personal probability distributions (ppd's) on the model space X to describe their prior beliefs about the location of $\mathbf{x}_E$ in X. Then they have used the classical theory of conditional probability (Bayesian calculus) to compute posterior ppd's for $\mathbf{x}_E$ from the data and the prior ppd's (Backus, 1970a, 1988a; Jackson, 1979; Gubbins, 1983; Tarantola, 1987). Subjectivists sometimes introduce a prior ppd that purports to be a probabilistic imitation of some generally accepted bound on $\mathbf{x}_E$. Backus (1988b) showed that such an imitation introduces spurious prior "information" about $\mathbf{x}_E$ that is not implied by the bound being imitated. Appendix A gives the details for gaussian probabilistic imitations of quadratic bounds.

In the present paper we take no position on the question of whether to interpret probability objectively or subjectively. We procrastinate on the philosophical issue, and leave it to each camp to interpret our final results according to their own views of probability.

We will need some of the notation used in the theories of sets, functions, measures and Hilbert spaces. Most of this is

in MacLane and Birkhoff (1967) and Halmos (1950, 1951, 1958). For completeness, we list it in Appendix B.

2. THE GENERAL INVERSE PROBLEM

We give here a concise description of a typical geophysical inverse problem. It provides us with D real numbers, $y^1, \cdots, y^D$ (the data). From the data we try to estimate P other real numbers $z^1, \cdots, z^P$ (the predictions). Both the data and the predictions could be calculated exactly from the correct earth model, $\mathbf{x}_E$, if we knew it. However, it is an unknown member of a known model space X, ususally infinite dimensional. We introduce two finite-dimensional real vector spaces, Y and Z, the data space and the prediction space, consisting respectively of all ordered D-tuples and all ordered P-tuples of real numbers. We are given a data function $F: X \to Y$ and a prediction function $G: X \to Z$, whose coordinate functions we denote by $F^i: X \to \mathbf{R}$ and $G^j: X \to \mathbf{R}$, where $1 \le i \le D$ and $1 \le j \le P$. If there were no errors, the observed data would be the entries $y_E^i = F^i(\mathbf{x}_E)$ in the D-tuple $\mathbf{y}_E = F(\mathbf{x}_E)$. We want to find the entries $z_E{}^j = G^j(\mathbf{x}_E)$ in the P-tuple $\mathbf{z}_E = G(\mathbf{x}_E)$. It is crucial to recognize the futility of trying to find $\mathbf{x}_E$ itself. We cannot record infinitely many real numbers, much less compute with them. Of course $\mathbf{z}_E$ may consist of P of the infinitely many parameters required for a complete description of $\mathbf{x}_E$.

The data include both a random error $\delta_R \mathbf{y}$ and a systematic error $\delta_S \mathbf{y}$, so that the measured data vector $\mathbf{y}_0$ is related to the error-free data vector $\mathbf{y}_E = F(\mathbf{x}_E)$ through the equation

$$\mathbf{y}_0 = \mathbf{y}_E + \delta_R \mathbf{y} + \delta_S \mathbf{y}. \tag{2.1}$$

By definition, an error is not known exactly, but if we know nothing about it then the data are useless. Systematic and random errors are distinguished by the kinds of partial information we have about them. For the systematic error $\delta_S \mathbf{y}$ we know only that

$$\delta_S \mathbf{y} \in V_S \tag{2.2}$$

where V_S is a known subset of the data space Y. We will call V_S the confinement set for $\delta_S \mathbf{y}$. Often Y has a norm, $\|\cdot\|$, and V_S is the solid ball $B_Y(\alpha)$ with radius α centered on $\mathbf{0}$ in Y. Then (2.2) becomes simply $\|\delta_S \mathbf{y}\| \le \alpha$. Extending Jackson's (1979) terminology, we will describe (2.2) as a "hard bound" on $\delta_S \mathbf{y}$.

A random error $\delta_R \mathbf{y}$ is a realization of a vector random variable taking values in Y. This random variable is completely described by its probability distribution, a probability measure η_R on Y. If V is any Borel subset of Y, $\eta_R(V)$ is the probability of the event $\delta_R \mathbf{y} \in V$. Following Jackson (1979) we call η_R a "soft bound" on $\delta_R \mathbf{y}$. We will assume that η_R is known except for a few parameters determined by fitting the data. Backus (1989) discusses some aspects of estimating those parameters. We will ignore that question here. We will assume that y^i and $y^i y^j$ are integrable with respect to η_R, and as a mnemonic device we will write $E(\eta_R, y^i)$ and $E(\eta_R, y^i y^j)$ in the forms $E(\delta_R y^i)$ and $E(\delta_R y^i \delta_R y^j)$. We will assume that

$$E(\delta_R y^i) = 0 \tag{2.3}$$

because a non-zero $E(\delta_R y^i)$ belongs in y_E^i if known and in $\delta_S y^i$ if unknown. Because of (2.3), the $D \times D$ variance matrix V of $\delta_R \mathbf{y}$ has the entries

$$V^{ij} = E(\delta_R y^i \delta_R y^j). \tag{2.4}$$

Given $\mathbf{y}_0$, the confinement set V_S for $\delta_S \mathbf{y}$ and the probability distribution η_R for $\delta_R \mathbf{y}$, we might try to predict the value of $\mathbf{z}_E = G(\mathbf{x}_E)$. Such an attempt is doomed unless $G = H \circ F$ for some function $H: Y \to Z$ (Backus and Gilbert, 1967). Usually there is no such H and doom is avoided by including in the inversion some prior information about $\mathbf{x}_E$, information available to us independently of $\mathbf{y}_0, V_S$ and η_R. This information usually takes the form of a hard bound or a soft bound on $\mathbf{x}_E$ in X. Some workers also admit "firm" bounds, hard bounds that are imprecisely known. We will not do so, because our prior information is often imprecise. Any inversion must include verifying that the results are insensitive to geophysically acceptable changes in the prior information.

In the problem of geomagnetic inference at the CMB the model space X consists of all magnetic fields whose sources lie inside the core of the earth. The data vector $\mathbf{y}$ consists of D elements of the geomagnetic field $\mathbf{B}$ measured at satellite altitudes or on the earth's surface. The prediction vector $\mathbf{z}_E$ might consist, for example, of P coefficients in the spherical harmonic expansion of the magnetic scalar potential. It could also include some fluxes through null flux curves on the core-mantle boundary (CMB). One example of a hard prior bound on the correct model $\mathbf{x}_E$ is that its magnetic energy must be less than minus the gravitational self energy of the earth (see appendix B). Another hard bound is that the ohmic heat production in the core must be less than the heat loss out of the earth's surface (Parker, 1972; Gubbins, 1983; Backus, 1988b). This latter bound might fail if the earth's thermal regime were significantly non-steady or if some of the ohmic core heat helped to drive the dynamo instead of diffusing out of the core (Backus, 1975).

In the geomagnetic problem, an example of a soft bound might be a probability distribution for the radial magnetic field on the CMB. Objectivists would want such a probability measure μ_E on X to have an objective basis, perhaps a well-accepted dynamo theory or perhaps simply an empirical fit to the data. For a subjectivist, μ_E could describe any personal beliefs. A probability distribution, however, usually cannot describe belief in a hard bound. Backus (1988a) showed that no hard bound on the norm of $\mathbf{x}_E$ can be approximated by a probability distribution. Appendix A

gives the details for the simple case of gaussian distributions. If the subjectivist starts out from a prior hard bound widely accepted by the community, and claims to believe the extra prior information implied by some probabilistic imitation of it, his inversion of the data will be no more convincing to colleagues than is the extra prior information.

Henceforth we will use (X,F,G) as the label for the inverse problem we have just described. This label is convenient but incomplete. It makes no mention of the error bounds or the prior information, both of which are essential parts of the problem. For example, the prior information about $\mathbf{x}_E$ often determines what properties of $\mathbf{x}_E$ can be estimated from the data (Backus, 1989). We will call the inverse problem (X,F,G) linear if X is a real vector space and F and G are linear functions. Whether (X,F,G) is linear or not, our goal is to find an estimate $\mathbf{z}_0$ for $\mathbf{z}_E$ and to describe the error in that estimate.

3. TRUNCATED APPROXIMATIONS

The present section describes a formal solution to the inverse problem (X,F,G) set forth in the preceding section. Some of the steps in this formalism involve delicate questions of functional and numerical analysis. The later parts of the present paper discuss those questions for linear problems, but for non-linear problems we offer only a description of what to compute, not how to compute it. Our formalism focuses on truncation and approximation, since we believe these are the crucial issues in modelling. If treated properly, they produce an inversion scheme which ought to satisfy both Bayesians and frequentists, and which can supply error estimates based on either hard or soft prior bounds.

The formalism depends on finding what we will call a truncated approximation (TA) to the inverse problem (X,F,G). A TA is an ordered quintuple $TA = (X_{TA},P_{TA},F_{TA},Q_{TA},G_{TA})$. The first entry of TA, X_{TA}, is a subset of a finite-dimensional topological space. It is often, but need not be, a finite-dimensional subspace of the model space X. The last four entries of TA are functions, whose domains and codomains are as follows:

$$P_{TA}: X \to X_{TA}\,, \tag{3.1a}$$

$$F_{TA}: X_{TA} \to Y\,, \tag{3.1b}$$

$$Q_{TA}: Y \to Y_{TA} \quad \text{where} \quad Y_{TA} = F_{TA}(X_{TA})\,, \tag{3.1c}$$

$$G_{TA}: X_{TA} \to Z\,. \tag{3.1d}$$

We put no restrictions on these functions except to demand that Q_{TA} be continuous, that

$$Q_{TA} \mid Y_{TA} = I_{Y_{TA}}\,, \tag{3.2a}$$

and that F_{TA} have a continuous inverse,

$$F_{TA}^{-1}: Y_{TA} \to X_{TA}\,. \tag{3.2b}$$

Note that (3.2a) implies

$$Q_{TA} \circ Q_{TA} = Q_{TA}\,. \tag{3.2c}$$

We call the TA $(X_{TA},P_{TA},F_{TA},Q_{TA},G_{TA})$ linear if (X,F,G) is a linear problem, X_{TA} is a linear space, and $P_{TA},F_{TA},Q_{TA},G_{TA}$ are linear functions.

Although the prior information about $\mathbf{x}_E$ is not explicitly listed in labelling the inverse problem as (X,F,G), that information usually plays a crucial role in constructing useful TA's. For example, if X is the space of magnetic fields $\mathbf{B}$ with sources only in the core and the data are intensities $|\mathbf{B}|$, then any field $\mathbf{B}$ that fits the data will have a twin, $-\mathbf{B}$, that also fits the data. Unless we have prior information about the direction of $\mathbf{B}$ (e.g. that the dipole moment is positive) we cannot construct an invertible F_{TA} or a TA. (This particular inverse problem has another, more serious, non-uniqueness whose remedy is at present unknown; Backus, 1970b.)

An inverse problem has many TA's. How to choose the best or even merely a good one can be discussed only after we have seen in the present section how to use them. We will treat optimization later and only for linear TA's. To use a TA we begin by defining its errors of truncated approximation,

$$\delta_{TA}\mathbf{y} = f_{TA}(\mathbf{x}_E), \quad \delta_{TA}\mathbf{z} = g_{TA}(\mathbf{x}_E) \tag{3.3a}$$

where

$$f_{TA} = F_{TA} \circ P_{TA} - F\,, \quad g_{TA} = G_{TA} \circ P_{TA} - G\,. \tag{3.3b}$$

These definitions are equivalent to the statements

$$\delta_{TA}\mathbf{y} = F_{TA}(P_{TA}(\mathbf{x}_E)) - F(\mathbf{x}_E), \quad \delta_{TA}\mathbf{z} = G_{TA}(P_{TA}(\mathbf{x}_E)) - G(\mathbf{x}_E).$$

If our prior information is a hard bound on $\mathbf{x}_E$, say

$$\mathbf{x}_E \in U_E\,, \tag{3.4a}$$

then from (3.3a) we have hard bounds for $\delta_{TA}\mathbf{y}$ and $\delta_{TA}\mathbf{z}$, namely

$$\delta_{TA}\mathbf{y} \in V_{TA} = f_{TA}(U_E), \quad \delta_{TA}\mathbf{z} \in W_{TA} = g_{TA}(U_E). \tag{3.4b}$$

If our prior information is a soft bound on $\mathbf{x}_E$, a probability measure (μ_E,Σ_X) for $\mathbf{x}_E$ in X, then $\mu_E: \Sigma_X \to \mathbf{R}$ and we have soft bounds on $\delta_{TA}\mathbf{y}$ and $\delta_{TA}\mathbf{z}$. These are probability measures (η_{TA},Σ_Y) and (ζ_{TA},Σ_Z) for $\delta_{TA}\mathbf{y}$ in Y and $\delta_{TA}\mathbf{z}$ in Z. They are defined as

$$\eta_{TA} = \mu_E \circ f_{TA}^{-1}\,, \quad \zeta_{TA} = \mu_E \circ g_{TA}^{-1}\,. \tag{3.5}$$

Since $\delta_{TA}\mathbf{y}$ and $\delta_{TA}\mathbf{z}$ are both functions of the random variable $\mathbf{x}_E$, they are themselves random variables but will not usually be independent.

If a TA, $(X_{TA},P_{TA},F_{TA},Q_{TA},G_{TA})$, is available, it provides a complete formal solution of the inverse problem (X,F,G). Since $\mathbf{y}_E = F(\mathbf{x}_E)$, (2.1) and (3.3) imply

$$F_{TA} \circ P_{TA}(\mathbf{x}_E) = \mathbf{y}_0 - \delta_R\mathbf{y} - \delta_S\mathbf{y} + \delta_{TA}\mathbf{y}\,.$$

By (3.2a), $Q_{TA} \circ F_{TA} = F_{TA}$, so

$$F_{TA} \circ P_{TA}(\mathbf{x}_E) = Q_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + \delta_{TA} \mathbf{y}).$$

But $F_{TA}^{-1}: Y_{TA} \to X_{TA}$ exists, so the foregoing equation implies

$$P_{TA}(\mathbf{x}_E) = F_{TA}^{-1} \circ Q_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + \delta_{TA} \mathbf{y}).$$

Therefore

$$G_{TA} \circ P_{TA}(\mathbf{x}_E) = G_{TA} \circ F_{TA}^{-1} \circ Q_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + \delta_{TA} \mathbf{y}).$$

Now $\mathbf{z}_E = G(\mathbf{x}_E)$, so (3.3) and the foregoing equation imply that

$$\mathbf{z}_E = H_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + \delta_{TA} \mathbf{y}) - \delta_{TA} \mathbf{z} \qquad (3.6a)$$

where

$$H_{TA} = G_{TA} \circ F_{TA}^{-1} \circ Q_{TA}. \qquad (3.6b)$$

We can rewrite these results as

$$\mathbf{z}_E = \mathbf{z}_0 - \delta \mathbf{z} \qquad (3.7a)$$

if we define

$$\mathbf{z}_0 = H_{TA}(\mathbf{y}_0) \qquad (3.7b)$$

and define

$$\delta \mathbf{z} = H_{TA}(\mathbf{y}_0) - H_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + \delta_{TA} \mathbf{y}) + \delta_{TA} \mathbf{z} \qquad (3.7c)$$

Equations (3.6) or (3.7) constitute a formal solution to the inverse problem (X, F, G). Under certain conditions that formal solution offers an opportunity to use the data to test the hypothesis that η_R is the correct probability measure for $\delta_R \mathbf{y}$. Those conditions are that Q_{TA} be linear and not surjective and that $\delta_{TA} \mathbf{y}$ and $\delta_S \mathbf{y}$ be negligible in comparison with $\delta_R \mathbf{y}$. To test the hypothesis, note that since $\mathbf{y}_E + \delta_{TA} \mathbf{y} = F_{TA} \circ P_{TA}(\mathbf{x}_E) = Q_{TA} \circ F_{TA} \circ P_{TA}(\mathbf{x}_E)$, therefore from (3.2c)

$$(I_Y - Q_{TA})(\mathbf{y}_E + \delta_{TA} \mathbf{y}) = \mathbf{0}. \qquad (3.8a)$$

Invoking (2.1) and the assumed linearity of Q_{TA}, we conclude that

$$(I_Y - Q_{TA})(\mathbf{y}_0) = (I_Y - Q_{TA})(\delta_R \mathbf{y} + \delta_S \mathbf{y} - \delta_{TA} \mathbf{y}). \qquad (3.8b)$$

If $\delta_S \mathbf{y}$ and $\delta_{TA} \mathbf{y}$ can be neglected in (3.8b), then the right hand side of that equation reduces to $(I_Y - Q_{TA})(\delta_R \mathbf{y})$. This latter vector is drawn at random from the subspace $(I_Y - Q_{TA})(Y)$ of Y, and its probability measure on that subspace should be

$$\eta_{RQ} = \eta_R \circ (I_Y - Q_{TA})^{-1} \qquad (3.9)$$

if η_R really describes $\delta_R \mathbf{y}$. Given η_R, we can calculate η_{RQ} from (3.9) and then use various tests (Kendall and Stuart II, 1979, Ch.30; see also Backus, 1989) to compute the probability that $(I_Y - Q_{TA})(\mathbf{y}_0)$ could have been drawn at random from the distribution (3.9).

It remains to understand and control the size of the error $\delta \mathbf{z}$ in (3.7c). For non-linear problems, such estimates can often be based on the modulus of continuity of H_{TA} and on how closely $F_{TA} \circ P_{TA}$ and $G_{TA} \circ P_{TA}$ resemble $Q_{TA} \circ F$ and G respectively. For linearizable problems we will later discuss these estimates in some detail.

The rest of this section examines how objectivists and subjectivists would react to (3.6) and (3.7). Objectivists have only two cases to consider: is the prior bound on $\mathbf{x}_E$ hard or soft? First, suppose it hard, as in (3.4a), so that $\delta_{TA} \mathbf{y}$ and $\delta_{TA} \mathbf{z}$ are systematic errors with confinement sets (3.4b). Objectivists would convert the soft bound on $\delta_R \mathbf{y}$ to a hard bound on $\mathbf{z}_E$ as follows: choose a small positive number ε_R and a Borel set V_R of Y as small as possible in some sense (perhaps in diameter if Y has a norm) but large enough to assure that $\eta_R(V_R) > 1 - \varepsilon_R$. If $\delta_R \mathbf{y} \in V_R$ then (3.6a) implies that

$$\mathbf{z}_E \in H_{TA}(\mathbf{y}_0 - V_R - V_S + V_{TA}) - W_{TA}. \qquad (3.10)$$

If (3.10) is false, then an event $(\delta_R \mathbf{y} \notin V_R)$ has occurred whose probability is less than ε_R. That is, (3.10) is true at confidence level at least $1 - \varepsilon_R$. If the data arise from aspects of $\mathbf{x}_E$ that are relevant to the predictions, then an adroit choice of a truncated approximation TA will make the confinement set (3.10) for $\mathbf{z}_E$ small enough to be interesting. In passing, we note that the confinement set (3.10) can be replaced by the smaller set consisting of all those $\mathbf{z} \in Z$ that can be written $H_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} - \delta_S \mathbf{y} + f_{TA}(\mathbf{x})) - g_{TA}(\mathbf{x})$ with $\delta_R \mathbf{y} \in V_R$, $\delta_S \mathbf{y} \in V_S$, and $\mathbf{x} \in U_E$.

For an objectivist the second case is that the prior bound on $\mathbf{x}_E$ is soft, a probability measure (μ_E, Σ_X) locating $\mathbf{x}_E$ in X. Now the objectivist chooses *two* small positive numbers, ε_R and ε_E and two subsets $V_R \subset Y$ and $U_E \subset X$, as small as possible in some sense but large enough that $\eta_R(V_R) > 1 - \varepsilon_R$ and $\mu_E(U_E) > 1 - \varepsilon_E$. The objectivist would use this U_E in (3.4b) to define V_{TA} and W_{TA}. If $\delta_R \mathbf{y} \in V_R$ and $\mathbf{x}_E \in U_E$, than (3.10) holds. Otherwise, either $\delta_R \mathbf{y} \notin V_R$ or $\mathbf{x}_E \notin U_E$ or both. The probability of this union of two events is at most the sum of their separate probabilities, $\varepsilon_R + \varepsilon_E$. Thus (3.10) is true at a confidence level of at least $1 - \varepsilon_R - \varepsilon_E$. Again, as in the preceding paragraph, the confinement set (3.10) can be slightly improved.

For objectivists it remains to say what "small" means when applied to V_R or U_E. If $dim\ Z = 1$, the answer is simple: for a given ε_R (or ε_R and ε_E) choose V_R (or V_R and U_E) so as to minimize the diameter of the set (3.10). Usually that set will be an interval, and then its length should be minimized. If $dim\ Z > 1$, the situation is less simple, and Stark (1992) and Donaho (1992) have discussed various optimizations in some detail.

Subjectivists will approach (3.6) and (3.7) differently, in order to exploit their view that they can describe their beliefs about $\mathbf{z}_E$ by means of a personal probability measure (ζ_E, Σ_Z) on Z. A naive subjectivist approach to (3.6) is to convert all the hard bounds to probability distributions and then to find ζ_E from standard probability calculus. This naive method fails because $\delta_S \mathbf{y}$ and $\mathbf{x}_E$ belong to spaces of

high dimension, so softening their hard bounds adds spurious "information" (Backus, 1988b; see also appendix A). The naive approach does succeed when the systematic errors are negligible in comparison with the random errors. We give two examples.

First, suppose $\delta_S \mathbf{y}$, $\delta_{TA} \mathbf{y}$, and $\delta_{TA} \mathbf{z}$ are all systematic errors, but are negligible in (3.6a). Then that equation can be written approximately as

$$\mathbf{z}_E = H_{TA}(\mathbf{y}_E) \text{ where } \mathbf{y}_E = \mathbf{y}_0 - \delta_R \mathbf{y}. \tag{3.11}$$

Subjectivists, unlike objectivists, would be willing to regard $\mathbf{y}_E$ as a random variable. In principle it can take only its true value, but a subjectivist would describe his personal opinion about it by means of a personal probability measure (η_E, Σ_Y) on Y. Most subjectivists would obtain this η_E from the error distribution η_R by requiring for each Borel subset V of Y that

$$\eta_E(V) = \eta_R(\mathbf{y}_0 - V). \tag{3.12a}$$

The argument for (3.12a) is that because of (3.11) the two events $\mathbf{y}_E \in V$ and $\delta_R \mathbf{y} \in \mathbf{y}_0 - V$ are the same, and so must have the same probability. Applying this argument to $\mathbf{z}_E$ in (3.11) would lead subjectivists to adopt

$$\zeta_E = \eta_E \circ H_{TA}^{-1} \tag{3.12b}$$

as the probabilistic description of their belief about where $\mathbf{z}_E$ lies in Z.

In the second subjectivist example without hard bounds, suppose that $\delta_S \mathbf{y}$ is negligible in (3.6a) and that the prior information is a soft bound, a probability measure (μ_E, Σ_X) on X. Using (3.3), we write (3.6a) as

$$\mathbf{z}_E = H_{TA}(\mathbf{y}_0 - \delta_R \mathbf{y} + f_{TA}(\mathbf{x}_E)) - g_{TA}(\mathbf{x}_E). \tag{3.13}$$

The right side of (3.13) is a function of $\mathbf{y}_0$, $\delta_R \mathbf{y}$ and $\mathbf{x}_E$, so we write it $h_{TA}(\mathbf{y}_0, \delta_R \mathbf{y}, \mathbf{x}_E)$. Then $h_{TA}(\mathbf{y}_0, \cdot, \cdot): Y \times X \to Z$. A subjectivist would likely regard $\delta_R \mathbf{y}$ and $\mathbf{x}_E$ as independent random variables, so their joint probability measure on $Y \times X$ would be the product measure $\eta_R \mu_E$ of η_R and μ_E (Halmos, 1950). Then a subjectivist would choose

$$\zeta_E = (\eta_R \mu_E) \circ h_{TA}(\mathbf{y}_0, \cdot, \cdot)^{-1}. \tag{3.14}$$

If neither random nor systematic errors are negligible and the inverse problem is nonlinear, we have no advice for subjectivists except to linearize (*vide infra*) or to become temporary objectivists.

If H_{TA} in (3.6) is linear or linearizable, error estimation from (3.7) is much simplified for both objectivists and subjectivists. If $\delta_R \mathbf{y}$, $\delta_S \mathbf{y}$ and $\delta_{TA} \mathbf{y}$ are small enough and DH_{TA} is the gradient function of H_{TA} at $\mathbf{y}_0$, then to first order in the errors we can write (3.7c) as

$$\delta \mathbf{z} = \Delta_{TA} \mathbf{z} + \delta_R \mathbf{z} + \delta_S \mathbf{z} \quad \text{where} \tag{3.15a}$$

$$\delta_R \mathbf{z} = DH_{TA}(\delta_R \mathbf{y}) \tag{3.15b}$$

$$\delta_S \mathbf{z} = DH_{TA}(\delta_S \mathbf{y}) \tag{3.15c}$$

$$\Delta_{TA} \mathbf{z} = (g_{TA} - DH_{TA} \circ f_{TA})(\mathbf{x}_E) \tag{3.15d}$$

Here f_{TA} and g_{TA} are given by (3.3b), and if H_{TA} is linear then $DH_{TA} = H_{TA}$. Clearly $\delta_S \mathbf{z}$ is a systematic error with confinement set $DH_{TA}(V_S)$, V_S being as in (2.2), and $\delta_R \mathbf{z}$ is a random error with probability measure

$$\zeta_R = \eta_R \circ (DH_{TA})^{-1}, \tag{3.16}$$

η_R being the probability measure for $\delta_R \mathbf{y}$. Finally, if the prior information about $\mathbf{x}_E$ is the hard bound (3.4a), then $\Delta_{TA} \mathbf{z}$ is a systematic error with confinement set $(g_{TA} - DH_{TA} \circ f_{TA})(U_E)$. If the prior information about $\mathbf{x}_E$ is a probability measure (μ_E, Σ_X), then $\Delta_{TA} \mathbf{z}$ is a random error with probability measure $\zeta_{TA} = \mu_E \circ (g_{TA} - DH_{TA} \circ f_{TA})^{-1}$.

When the linearization (3.15) is possible, if we estimate $\mathbf{z}_E$ as $H_{TA}(\mathbf{y}_0)$, we commit an error $\delta \mathbf{z}$ that is the sum of one random and one systematic error. If the prior bound on $\mathbf{x}_E$ is hard, the random part of $\delta \mathbf{z}$ is $\delta_R \mathbf{z}$ and the systematic part is $\Delta_{TA} \mathbf{z} + \delta_S \mathbf{z}$. The confinement set of the latter is $(g_{TA} - DH_{TA} \circ f_{TA})(U_E) + DH_{TA}(V_S)$. If the prior bound on $\mathbf{x}_E$ is soft, the systematic part of $\delta \mathbf{z}$ is $\delta_S \mathbf{z}$ and the random part is $\Delta_{TA} \mathbf{z} + \delta_R \mathbf{z}$. It seems reasonable to suppose that $\Delta_{TA} \mathbf{z}$ and $\delta_R \mathbf{z}$ are independent, so the probability measure of their sum is the convolution of their probability measures (Kendall and Stuart, 1977, p200ff). When H_{TA} is linearizable, the simple form of $\delta \mathbf{z}$ as the sum of a random and a systematic error somewhat simplifies the objectivist's calculation of a confidence set. Linearizability has a more profound effect for the subjectivist. If $dim\ Z \geq 3$, appendix A suggests that subjectivists seeking a wide audience will want to report the error in $\delta \mathbf{z}$ as the sum of two errors, a random error with known probability measure and a systematic error with known confinement set. Imitating the systematic error with a subjective probability measure when $dim\ Z \geq 3$ will introduce spurious information. Only if $dim\ Z \leq 2$ will subjectivists want to soften the confinement set for the systematic error in $\mathbf{z}$ to a probability distribution and obtain a probability measure for $\delta \mathbf{z}$ as the convolution of two random parts. Only when H_{TA} is linearizable and subjectivists have softened the systematic error in $\mathbf{z}$ can they speak sensibly of the correlation between the errors in two predictions. Objectivists would never do so except when all systematic errors were negligible.

4. NATURAL DOT PRODUCTS ON X, X_{TA} AND Y

In every inverse problem the probability measure η_R for the random error in the data vector generates a dot product on the data space Y. In certain inverse problems the prior information about the correct earth model $\mathbf{x}_E$ generates a dot product on the model space X or the truncated space X_{TA}. When available, these dot products are powerful tools for estimating the prediction error (3.15). Therefore, we briefly examine all three dot products, following Backus (1989),

who discussed two of them at length and derived the machinery for obtaining the third.

We begin with the dot product on X. This is available when X is a real vector space and the prior information is a quadratic bound for the correct earth model $\mathbf{x}_E$:

$$Q(\mathbf{x}_E, \mathbf{x}_E) \le q. \tag{4.1}$$

Here q is a known positive real number and Q is a known symmetric, positive-definite bilinear form on X. That is, for any $\mathbf{x}_1, \mathbf{x}_2 \in X$, $Q(\mathbf{x}_1, \mathbf{x}_2)$ is a real number that depends linearly on each of $\mathbf{x}_1, \mathbf{x}_2$ when the other is fixed; also $Q(\mathbf{x}_1, \mathbf{x}_2) = Q(\mathbf{x}_2, \mathbf{x}_1)$ and, if $\mathbf{x} \neq \mathbf{0}$, then $Q(\mathbf{x}, \mathbf{x}) > 0$. Given a quadratic bound (4.1), we define the dot product of any $\mathbf{x}_1, \mathbf{x}_2 \in X$ to be

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = q^{-1} Q(\mathbf{x}_1, \mathbf{x}_2). \tag{4.2}$$

We write the norm for this dot product as $\| \mathbf{x} \|$, so

$$\| \mathbf{x} \| = (\mathbf{x} \cdot \mathbf{x})^{1/2}. \tag{4.3}$$

In terms of this norm, the prior quadratic bound on $\mathbf{x}_E$ is simply

$$\| \mathbf{x}_E \| \le 1. \tag{4.4}$$

If X is not complete in the norm (4.3), we complete it, so that it becomes a real Hilbert space with dot product (4.2) (Halmos, 1951). If $dim\, X$ were finite, this step would be unnecessary; finite dimensional spaces are always complete, i.e. Hilbert spaces, and all their subspaces are closed. Every closed subspace U of X has an orthogonal projector $\Pi_U : X \to U$.

Next we consider the dot product generated on X_{TA} . This dot product is available when TA is a truncated approximation to (X, F, G), when X_{TA} is a real vector space, and when the prior information about $\mathbf{x}_E$ is a probability measure μ_E on X. On X_{TA} we define a probability measure μ_{TA} as

$$\mu_{TA} = \mu_E \circ P_{TA}^{-1}. \tag{4.5}$$

Backus (1989) shows that, since X_{TA} is finite-dimensional, there is a unique dot product on it relative to which the variance tensor of μ_{TA} is the identity tensor. To define this dot product without invoking tensors, we note that it is uniquely determined by the fact that for any fixed $\mathbf{w}_1$ and $\mathbf{w}_2$ in X_{TA}

$$E[(\mathbf{w}_1 \cdot \mathbf{w})(\mathbf{w}_2 \cdot \mathbf{w})] - E[\mathbf{w}_1 \cdot \mathbf{w}] E[\mathbf{w}_2 \cdot \mathbf{w}] = \mathbf{w}_1 \cdot \mathbf{w}_2, \tag{4.6}$$

where $\mathbf{w}$ is a vector random variable in X_{TA} with probability measure μ_{TA} (see Backus, 1989).

Finite dimensionality is crucial to the foregoing construction of a dot product on X_{TA} . Backus (1988b) showed that this construction must fail in an infinite dimensional space because then the part of the space consisting of vectors with finite norm has measure zero.

Finally we turn to the dot product on Y, which we construct from the $D \times D$ error variance matrix V defined by (2.4). We adopt the Einstein summation convention: in a term or product of terms if an index appears once as a subscript and once as a superscript, then a sum over all possible values of the index is understood. We will assume that no linear combination of $y^1, \cdots, y^D$ can be measured with perfect accuracy. This means that if $a_1, \cdots, a_D$ are any real numbers not all zero, then $0 < E[(a_i \delta_R y^i)^2] = E[(a_i \delta_R y^i)\ (a_j \delta_R y^j)] = a_i a_j E[\delta_R y^i\ \delta_R y^j] = a_i a_j V^{ij}$. Therefore the matrix V of (2.4) is positive definite. It is obviously symmetric, so it has a positive-definite symmetric inverse, V^{-1}.

If η_R, the probability measure for $\delta_R \mathbf{y}$, were gaussian, its density function would be a constant times $\exp(-y^i V_{ij}^{-1} y^j / 2)$. This suggests introducing the following dot product on Y even when η_R is not gaussian: if $\mathbf{u} = (u^1, \cdots, u^D)$ and $\mathbf{v} = (v^1, \cdots, v^D)$ then

$$\mathbf{u} \cdot \mathbf{v} = u^i V_{ij}^{-1} v^j . \tag{4.7}$$

The fact that V^{-1} is real, symmetric and positive definite assures that (4.7) does define a dot product. We define the norm $\| \mathbf{y} \|$ for this dot product as in (4.3). Since $dim\, Y < \infty$, Y is complete in this norm, and every subspace U of Y has an orthogonal projector $\Pi_U : Y \to U$.

It seems intuitively obvious that increasing the random error $\delta_R \mathbf{y}$ decreases the norm of the data vector $\mathbf{y}$, since the latter is measured in units of the former. Later we will need this fact, so here we give a proof. Suppose that $\delta_R \mathbf{y} = \delta_R \mathbf{y}_1 + \delta_R \mathbf{y}_2$ where $\delta_R \mathbf{y}_1$ and $\delta_R \mathbf{y}_2$ are independent random variables with mean $\mathbf{0}$ and variance matrices V_1 and V_2. If V is the variance matrix of $\delta_R \mathbf{y}$, then, of course, $V = V_1 + V_2$. If $\| \mathbf{y} \|$ and $\| \mathbf{y} \|_1$ are the norms on Y produced by V and V_1, we claim that

$$\| \mathbf{y} \| < \| \mathbf{y} \|_1 \text{ for all nonzero } \mathbf{y} \in Y. \tag{4.8}$$

We need to show that if $\mathbf{y} \in Y$ and $\mathbf{y} \neq \mathbf{0}$ then $\mathbf{y} V^{-1} \mathbf{y}^T < \mathbf{y} V_1^{-1} \mathbf{y}^T$, or

$$\mathbf{y} (V_1^{-1} - V^{-1})\, \mathbf{y}^T > 0. \tag{4.9}$$

Here $\mathbf{y}^T$ is the column vector obtained by transposing the row vector $\mathbf{y}$. Every positive definite symmetric (pds) matrix has a pds inverse and a unique pds square root, so we can define the $D \times D$ pds matrix $A = V_1^{-1/2} V V_1^{-1/2}$. For any nonzero vector $\mathbf{y} \in Y$, we set $\mathbf{u} = \mathbf{y} V_1^{-1/2} A^{-1/2} V_1^{-1/2}$. Then a simple calculation shows that $\mathbf{y} (V_1^{-1} - V^{-1}) \mathbf{y}^T = \mathbf{u} (V - V_1) \mathbf{u}^T$. But $V - V_1 = V_2$. Since V_2 is positive definite and $\mathbf{u} \neq \mathbf{0}$, we have (4.9).

Perhaps the most important property of the dot product (4.7) is the analogue of (4.6): if $\mathbf{u}, \mathbf{v} \in Y$ then

$$E[(\mathbf{u} \cdot \delta_R \mathbf{y})(\mathbf{v} \cdot \delta_R \mathbf{y})] = \mathbf{u} \cdot \mathbf{v}. \tag{4.10}$$

In the language of Cartesian tensors (Jeffreys, 1969) (4.10) says that under the dot product (4.7) the expected value of the dyad $\delta_R \mathbf{y}\, \delta_R \mathbf{y}$ is the identity tensor on Y. To prove (4.10) without reference to tensors, note that

$(\mathbf{u}\cdot\delta_R\mathbf{y})(\mathbf{v}\cdot\delta_R\mathbf{y}) = (u^i V_{ij}^{-1}\delta_R y^j)(\delta_R y^k V_{kl}^{-1} v^l)$, so $E[(\mathbf{u}\cdot\delta_R\mathbf{y})(\mathbf{v}\cdot\delta_R\mathbf{y})] = u^i V_{ij}^{-1} E(\delta_R y^j\,\delta_R y^k) V_{kl}^{-1} v^l = u^i V_{ij}^{-1} V^{jk} V_{kl}^{-1} v^l = u^i V_{il}^{-1} v^l = \mathbf{u}\cdot\mathbf{v}$.

5. DATA COMPRESSION

When the data are very numerous, it is common practice to combine them in various ways so as to generate a data vector $\tilde{\mathbf{y}}$ of lower dimension with smaller variance. The new data will be $\tilde{y}^1 = \tilde{K}^1(\mathbf{y}), \cdots, \tilde{y}^M = \tilde{K}^M(\mathbf{y})$ where $\tilde{K}^j : Y \to \mathbf{R}$ for $1 \le j \le M$. In this paper we will consider only *linear* functionals $\tilde{K}^j$. It is pointless to include among them one that is a linear combination of the others, so we will assume that $\{\tilde{K}^1, \cdots, \tilde{K}^M\}$ is linearly independent. Therefore $M \le D$. We will call the process of computing $\tilde{\mathbf{y}} = (\tilde{y}^1, \cdots, \tilde{y}^M)$ from $\mathbf{y} = (y^1, \cdots, y^D)$ a "compression" of the data. Unless $M = D$ the compression is irreversible; $\mathbf{y}$ cannot be recovered from $\tilde{\mathbf{y}}$. We let $\tilde{Y}$ be the vector space of real M-tuples, and we define the compression function $\tilde{K} : Y \to \tilde{Y}$ by

$$\tilde{K}(\mathbf{y}) = (\tilde{K}^1(\mathbf{y}), \cdots, \tilde{K}^M(\mathbf{y})). \tag{5.1}$$

From the original inverse problem (X, F, G) the compression $\tilde{K}$ produces a new, compressed inverse problem $(X, \tilde{F}, G)$ whose data space is $\tilde{Y}$. In this new problem the original data function F, the original probability measure η_R on Y for the random error $\delta_R\mathbf{y}$, and the original confinement set V_S for the systematic error $\delta_S\mathbf{y}$ are replace by the new, "compressed" values

$$\tilde{F} = \tilde{K} \circ F, \quad \tilde{\eta}_R = \eta_R \circ \tilde{K}^{-1}, \quad \tilde{V}_S = \tilde{K}(V_S). \tag{5.2a}$$

If $(X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$ is a truncated approximation of (X, F, G) and

$$\tilde{F}_{TA} = \tilde{K} \circ F_{TA}, \quad \tilde{Q}_{TA} = \tilde{K} \circ Q_{TA} \tag{5.2b}$$

then $(X_{TA}, P_{TA}, \tilde{F}_{TA}, \tilde{Q}_{TA}, G_{TA})$ is a truncated approximation of $(X, \tilde{F}, G)$.

The random errors in the compressed data are $\delta_R\tilde{y}^j = \tilde{K}^j(\delta_R\mathbf{y})$, and we can imitate (2.4) by defining

$$\tilde{V}^{ij} = E(\delta_R\tilde{y}^i\ \delta_R\tilde{y}^j), \tag{5.3}$$

where now $\tilde{V}$ is an $M \times M$ positive definite symmetric matrix. Then, imitating (4.5), we can define a dot product on $\tilde{Y}$. If $\tilde{\mathbf{u}} = (\tilde{u}^1, \cdots, \tilde{u}^M)$ and $\tilde{\mathbf{v}} = (\tilde{v}^1, \cdots, \tilde{v}^M)$ are any vectors in $\tilde{Y}$, then their dot product is

$$\tilde{\mathbf{u}} \tilde{\cdot} \tilde{\mathbf{v}} = \tilde{u}^i \tilde{V}_{ij}^{-1} \tilde{v}^j. \tag{5.4}$$

To use $\tilde{K}$ we must understand how the dot product (5.4) on $\tilde{Y}$ is related to the dot product (4.5) on Y. The answer hinges on dual sequences. Since Y is a finite-dimensional dot product space and $\tilde{K}^j : Y \to \mathbf{R}$ is a linear functional on Y, there is a unique $\mathbf{K}^j \in Y$ such that

$$\tilde{K}^j(\mathbf{y}) = \mathbf{K}^j \cdot \mathbf{y} \quad \text{for all} \quad \mathbf{y} \in Y \tag{5.5}$$

(Halmos, 1958). Define

$$U = span\,\{\mathbf{K}^1, \cdots, \mathbf{K}^M\}, \tag{5.6a}$$

$$U_j = span\,[\{\mathbf{K}^1, \cdots, \mathbf{K}^M\} \setminus \{\mathbf{K}^j\}]. \tag{5.6b}$$

Since $\{\mathbf{K}^1, \cdots, \mathbf{K}^M\}$ is linearly independent, $dim\ U = M$ and $dim\ U_j = M - 1$. Therefore $dim\,(U_j^{\perp} \cap U) = 1$, so $U_j^{\perp} \cap U$ contains exactly one vector whose dot product with $\mathbf{K}^j$ is 1. Call this vector $\mathbf{K}_j$. The ordered sequence $(\mathbf{K}_1, \cdots, \mathbf{K}_M)$ is the "dual sequence" of $(\mathbf{K}^1, \cdots, \mathbf{K}^M)$. It satisfies and is uniquely determined by two conditions:

$$\mathbf{K}_j \in U \quad \text{for} \quad 1 \le j \le M \tag{5.7a}$$

$$\mathbf{K}_i \cdot \mathbf{K}^j = \delta_i{}^j. \tag{5.7b}$$

(Hereafter $\delta_i{}^j$, $\delta^i{}_j$, δ^{ij} and δ_{ij} all denote the Kronecker delta, 1 if $i = j$ and 0 if $i \ne j$.) The fact that (5.7) uniquely determine the dual sequence shows that the dual sequence of $(\mathbf{K}_1, \cdots, \mathbf{K}_M)$ is $(\mathbf{K}^1, \cdots, \mathbf{K}^M)$. If $(\mathbf{K}^1, \cdots, \mathbf{K}^M)$ is orthonormal, then $\mathbf{K}^i \cdot \mathbf{K}^j = \delta^{ij}$, so uniqueness in (5.7) shows that an orthonormal sequence is self-dual.

Both $\{\mathbf{K}^1, \cdots, \mathbf{K}^M\}$ and $\{\mathbf{K}_1, \cdots, \mathbf{K}_M\}$ are bases for U. Therefore, from (5.7) it follows that if $\mathbf{u} \in U$ then

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{K}_j)\mathbf{K}^j = (\mathbf{u}\cdot\mathbf{K}^j)\mathbf{K}_j. \tag{5.8a}$$

If $\mathbf{v} \in Y$, then dotting $\mathbf{v}$ into (5.8a) gives

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{u}\cdot\mathbf{K}_j)(\mathbf{K}^j\cdot\mathbf{v}) = (\mathbf{u}\cdot\mathbf{K}^j)(\mathbf{K}_j\cdot\mathbf{v}). \tag{5.8b}$$

By symmetry, (5.8b) is also true if $\mathbf{u} \in Y$ and $\mathbf{v} \in U$. If we set $\mathbf{u} = \mathbf{K}_i$ and $\mathbf{v} = \mathbf{K}^k$ in (5.8b), then (5.7b) implies

$$(\mathbf{K}_i\cdot\mathbf{K}_j)(\mathbf{K}^j\cdot\mathbf{K}^k) = \delta_i{}^k. \tag{5.9a}$$

If we set $\mathbf{u} = \mathbf{K}^i$ and $\mathbf{v} = \mathbf{K}_k$ in (5.8b), then (5.7b) implies

$$(\mathbf{K}^i\cdot\mathbf{K}^j)(\mathbf{K}_j\cdot\mathbf{K}_k) = \delta^i{}_k. \tag{5.9b}$$

Therefore the two $M \times M$ matrices whose entries are $\mathbf{K}_i \cdot \mathbf{K}_j$ and $\mathbf{K}^i \cdot \mathbf{K}^j$ are inverse to one another.

It follows from (5.8a) that $\Pi_U : Y \to U$, the orthogonal projector of Y onto U, can be calculated in several ways:

$$\Pi_U(\mathbf{y}) = (\mathbf{y}\cdot\mathbf{K}_j)\,\mathbf{K}^j = (\mathbf{y}\cdot\mathbf{K}^j)\,\mathbf{K}_j = \tilde{K}^i(\mathbf{y})\,\mathbf{K}_j. \tag{5.10}$$

Now we can describe the effect of data compression on the dot product. This will involve showing that

$$\tilde{K}(U^{\perp}) = 0 \tag{5.11}$$

and that $\tilde{K}\,|\,U$ is an isometry between U with dot product (4.5) and $\tilde{Y}$ with dot product (5.4). That is, $\tilde{K}\,|\,U : U \to \tilde{Y}$ is a bijection for which, if $\mathbf{u}, \mathbf{v} \in U$, then

$$\tilde{K}(\mathbf{u}) \tilde{\cdot} \tilde{K}(\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}. \tag{5.12}$$

To prove (5.11), we observe from (5.10) that for $1 \le j \le M$ we have $\tilde{K}^i(P I_U(\mathbf{y})) = \mathbf{K}^i \cdot \Pi_U(\mathbf{y}) = \tilde{K}^j(\mathbf{y})(\mathbf{K}^i\cdot\mathbf{K}_j) = \tilde{K}^j(\mathbf{y})\,\delta^i{}_j = \tilde{K}^i(\mathbf{y})$, so $\tilde{K}(P I_U(\mathbf{y})) = \tilde{K}(\mathbf{y})$. Since this is true for all $\mathbf{y} \in Y$,

$$\tilde{K} \circ \Pi_U = \tilde{K}, \tag{5.13}$$

from which (5.11) follows immediately.

To prove (5.12), we note that $\tilde{K}(\mathbf{u}) = (\tilde{K}^1(\mathbf{u}), \cdots, \tilde{K}^M(\mathbf{u})) = (\mathbf{K}^1 \cdot \mathbf{u}, \cdots, \mathbf{K}^M \cdot \mathbf{u})$ and similarly for $\tilde{K}(\mathbf{v})$. Then by (5.4)

$$\tilde{K}(\mathbf{u}) \tilde{\cdot} \tilde{K}(\mathbf{v}) = (\mathbf{K}^i \cdot \mathbf{u})\, \tilde{V}_{ij}^{-1}\, (\mathbf{K}^j \cdot \mathbf{v}). \tag{5.14}$$

From (5.3) and (5.5) $\tilde{V}^{ij} = E[(\mathbf{K}^i \cdot \delta_R \mathbf{y})(\mathbf{K}^j \cdot \delta_R \mathbf{y})]$. Therefore, by (4.8)

$$\tilde{V}^{ij} = \mathbf{K}^i \cdot \mathbf{K}^j. \tag{5.15a}$$

It follows from (5.9) that

$$\tilde{V}_{ij}^{-1} = \mathbf{K}_i \cdot \mathbf{K}_j. \tag{5.15b}$$

Thus

$$\tilde{K}(\mathbf{u}) \tilde{\cdot} \tilde{K}(\mathbf{v}) = (\mathbf{u} \cdot \mathbf{K}^i)(\mathbf{K}_i \cdot \mathbf{K}_j)(\mathbf{K}^j \cdot \mathbf{v}) =$$

$$= [(\mathbf{u} \cdot \mathbf{K}^i)\mathbf{K}_i] \cdot [\mathbf{K}_j\, (\mathbf{K}^j \cdot \mathbf{v})].$$

Since $\mathbf{u}, \mathbf{v} \in U$, (5.8b) shows that this last term is $\mathbf{u} \cdot \mathbf{v}$. Now we have proved (5.12). But if $\tilde{K}(\mathbf{u}) = \mathbf{0}$ and $\mathbf{u} \in U$ then (5.12) implies that $\mathbf{u} = \mathbf{0}$. Therefore, $\tilde{K} \mid U : U \to \tilde{Y}$ being linear, it is an injection. Then it is a surjection because *dim* U = *dim* $\tilde{Y}$.

One way to interpret (5.11) and (5.12) is as follows: when we compress a data vector $\mathbf{y} = (y^1, \cdots, y^D)$ to a vector $\tilde{\mathbf{y}} = (\tilde{y}^1, \cdots, \tilde{y}^M)$ where $\tilde{y}^i = \tilde{K}^i(\mathbf{y}) = \mathbf{K}^i \cdot \mathbf{y}$, we lose all the information in $\Pi_U{}^{\perp}(\mathbf{y})$. The compressed data vector $\tilde{\mathbf{y}}$ can be thought of as a vector in U, namely $\tilde{y}^i\, \mathbf{K}_i$. In this way, every compressed data vector "is" a vector in U and *vice versa*.

Because of (5.12) and (5.13), we note finally that

$$\| \tilde{K}(\mathbf{y}) \| \le \| \mathbf{y} \| \quad \text{if} \quad \mathbf{y} \in Y. \tag{5.16}$$

6. TRUNCATORS FOR LINEAR INVERSE PROBLEMS WITH DOT PRODUCTS

In this section we suppose that the inverse problem is linear, that the the model space X is a Hilbert space, and that the data space Y has been provided with a dot product as in section 4. We also suppose that the data function F and the prediction function G are continuous. A discontinuous linear function on X is unbounded (Halmos, 1951), so arbitrarily small changes in its argument can produce arbitrarily large changes in its value. If F or G is discontinuous, then a stronger prior quadratic bound for x_E is needed (Backus, 1989) for a sensible inversion.

In the class of problems just described, there is a tool ready to hand for constructing a large class of truncated approximations. We call this tool a truncator. It is a function Ξ whose domain is the set of pairs (α,β) of real numbers such that

$$0 < \alpha < \beta. \tag{6.1a}$$

For any such pair, the function value $\Xi(\alpha,\beta)$ is a finite dimensional subspace of X with these properties:

$$\|F(\mathbf{x})\| \ge \alpha \|\mathbf{x}\| \quad \text{if} \quad \mathbf{x} \in \Xi(\alpha,\beta), \tag{6.1b}$$

$$\|F(\mathbf{x})\| \le \beta \|\mathbf{x}\| \quad \text{if} \quad \mathbf{x} \in \Xi(\alpha,\beta)^{\perp}. \tag{6.1c}$$

We show first how to construct a truncator and then how to obtain truncated approximations from it.

We begin by noting that a truncator always exists for a linear inverse problem (X,F,G) in which X is a Hilbert space and Y is a finite dimensional dot product space. Since $F : X \to Y$ is continuous, it is bounded (Halmos, 1951). That is, there is a number M such that $\|F(\mathbf{x})\| \le M \|\mathbf{x}\|$ for all $\mathbf{x} \in X$. The smallest such M is written $\|F\|$ and called the bound of F. By definition, $\|F\|$ is the smallest real number such that

$$\|F(\mathbf{x})\| \le \|F\| \|\mathbf{x}\| \quad \text{for all} \quad \mathbf{x} \in X. \tag{6.2}$$

Since F is bounded and Y is finite dimensional, F has a singular value decomposition (Backus, 1989). That is, there are orthonormal bases $\{\mathbf{x}_1, \mathbf{x}_2, \cdots\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_D\}$ for X and Y and non-negative real numbers $\lambda_1, \cdots, \lambda_D$ such that

$$F(\hat{\mathbf{x}}_n) = \lambda_n \hat{\mathbf{y}}_n \quad \text{if} \quad 1 \le n \le D, \tag{6.3a}$$

$$F(\hat{\mathbf{x}}_n) = \mathbf{0} \quad \text{if} \quad n > D. \tag{6.3b}$$

(Note that on the right in (6.3a) there is no implied sum on n because in both its appearances n is a subscript.) If $\alpha > \lambda_n$ for all n, we define $\Xi(\alpha,\beta)$ to be the space containing only $\mathbf{0}$. Otherwise, we define $\Xi(\alpha,\beta)$ to be the subspace of X spanned by those $\mathbf{x}_n$ for which $\lambda_n \ge \alpha$. To verify (6.1), note that if $\mathbf{x} \in X$ then

$$\mathbf{x} = \sum_{n=1}^{\infty} (\mathbf{x} \cdot \hat{\mathbf{x}}_n) \hat{\mathbf{x}}_n \qquad \text{so} \tag{6.4a}$$

$$F(\mathbf{x}) = \sum_{n=1}^{D} (\mathbf{x} \cdot \hat{\mathbf{x}}_n) \lambda_n \hat{\mathbf{y}}_n \qquad \text{and} \tag{6.4b}$$

$$\|F(\mathbf{x})\|^2 = \sum_{n=1}^{D} \lambda_n^2 (\mathbf{x} \cdot \hat{\mathbf{x}}_n)^2. \tag{6.4c}$$

If $\mathbf{x} \in \Xi(\alpha,\beta)$, then $\mathbf{x} \cdot \hat{\mathbf{x}}_n = 0$ if either $n > D$ or $\lambda_n < \alpha$, so

$$\|F(\mathbf{x})\|^2 \ge \alpha^2 \sum_{n=1}^{D} (\mathbf{x} \cdot \hat{\mathbf{x}}_n)^2 = \alpha^2 \|\mathbf{x}\|^2. \tag{6.4d}$$

Hence $\Xi(\alpha,\beta)$ satisfies (6.1b). For (6.1c), suppose $\mathbf{x} \in \Xi(\alpha,\beta)^{\perp}$. Then $\mathbf{x} \cdot \hat{\mathbf{x}}_n = 0$ if $n \le D$ and $\lambda_n \ge \alpha$, so

$$\|F(\mathbf{x})\|^2 < \alpha^2 \sum_{n=1}^{D} (\mathbf{x} \cdot \hat{\mathbf{x}}_n)^2 \le \alpha^2 \|\mathbf{x}\|^2. \tag{6.4e}$$

Since $\alpha < \beta$, $\Xi(\alpha,\beta)$ satisfies (6.1c).

The purpose of a truncated approximation of (X,F,G) is to make that inverse problem amenable to computation. The purpose of a truncator is to provide truncated approximations. The truncator described in the preceding paragraph

is an inauspicious beginning for this enterprise, since constructing it requires computing the singular value decomposition of an $\infty \times D$ matrix. In practice, of course, such an infinite process requires some sort of analytic justification of a finite approximation. In many inverse problems. our knowledge of the data function F provides such analytic information in the form of a "pre-truncator", a tool for constructing truncators. A pre-truncator is a function Π whose domain is the set of all positive real numbers. For any real $\gamma > 0$, $\Pi(\gamma)$ is a finite dimensional subspace of X such that

$$\|F(\mathbf{x})\| \le \gamma \|\mathbf{x}\| \quad \text{if} \quad \mathbf{x} \in \Pi(\gamma)^{\perp}. \tag{6.5}$$

That pre-truncators exist is trivial: if Ξ is a truncator and $\Pi(\gamma) = \Xi(\gamma/2, \gamma)$, then Π is a pre-truncator. But the idea is to go the other way. We hope to use our knowledge of F to find a pre-truncator analytically. We cannot say more about this process here, since it depends on the details of the F being studied, and success is not guaranteed. Once we have a pre-truncator, constructing a truncator from it is a finite computational task.

To see this, suppose that analysis of F has given us a pre-truncator Π. To construct a truncator from it, we note that for any $\gamma > 0$, $dim\ \Pi(\gamma) < \infty$, so there are efficient algorithms that provide a singular value decomposition of $F|\Pi(\gamma)$ (Golub and Van Loan, 1983; Press et al, 1992). From such a computation we obtain orthonormal bases $\{\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{N(\gamma)}\}$ and $\{\hat{\mathbf{y}}_1, \cdots, \hat{\mathbf{y}}_D\}$ for $\Pi(\gamma)$ and Y and non-negative real numbers $\lambda_1, \cdots, \lambda_{min[D,N(\gamma)]}$ such that

$$F(\hat{\mathbf{x}}_n) = \lambda_n \hat{\mathbf{y}}_n \quad \text{if} \quad 1 \le n \le min[D, N(\gamma)], \tag{6.6a}$$

$$F(\hat{\mathbf{x}}_n) = \mathbf{0} \quad \text{if} \quad min[D, N(\gamma)] < n \le N(\gamma). \tag{6.6b}$$

The $\hat{\mathbf{x}}_n$ and $\hat{\mathbf{y}}_n$ in (6.6) will usually differ from those in (6.3). Now we construct from Π a truncator Ξ as follows: given any α and β with $0 < \alpha < \beta$, choose γ so that

$$\beta = (\alpha^2 + \gamma^2)^{1/2}. \tag{6.7}$$

Compute the singular value decomposition (6.6) of the analytically constructed space $\Pi(\gamma)$. If $\alpha > \lambda_n$ for all n, let $\Xi(\alpha, \beta) = \{0\}$. Otherwise, let $\Xi(\alpha, \beta)$ be the subspace of $\Pi(\gamma)$ spanned by those $\hat{\mathbf{x}}_n$ in (6.6) for which $\lambda_n \ge \alpha$.

To prove that the Ξ so defined is a truncator, we appeal to (6.4). Those equations apply to the present situation if in (6.4a) we replace ∞ by N and in (6.4b-e) we replace D by $min[D, N(\gamma)]$. Therefore (6.1b) follows from (6.4d). It remains to prove (6.1c). To that end, suppose $\mathbf{x} \in \Xi(\alpha, \beta)^{\perp}$. Then we can write $\mathbf{x} = \mathbf{x}_\gamma + \mathbf{x}_\gamma^{\perp}$ where $\mathbf{x}_\gamma \in \Xi(\alpha, \beta)^{\perp} \cap \Pi(\gamma)$ and $\mathbf{x}_\gamma^{\perp} \in \Pi(\gamma)^{\perp}$. It follows from (6.4e) that

$$\|F(\mathbf{x}_\gamma)\| \le \alpha \|\mathbf{x}_\gamma\|$$

and from (6.5) that

$$\|F(\mathbf{x}_\gamma^{\perp})\| \le \gamma \|\mathbf{x}_\gamma^{\perp}\|.$$

But $F(\mathbf{x}) = F(\mathbf{x}_\gamma) + F(\mathbf{x}_\gamma^{\perp})$ so, by the triangle inequality, $\|F(\mathbf{x})\|$

$\le \|F(\mathbf{x}_\gamma)\| + \|F(\mathbf{x}_\gamma^{\perp})\|$. Therefore

$$\|F(\mathbf{x})\| \le \alpha \|\mathbf{x}_\gamma\| + \gamma \|\mathbf{x}_\gamma^{\perp}\| \quad \text{if} \quad \mathbf{x} = \mathbf{x}_\gamma + \mathbf{x}_\gamma^{\perp}. \tag{6.8a}$$

Since $\mathbf{x}_\gamma \cdot \mathbf{x}_\gamma^{\perp} = 0$, we have

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_\gamma\|^2 + \|\mathbf{x}_\gamma^{\perp}\|^2. \tag{6.8b}$$

Under the constraint (6.8b), the right side of (6.8a) has maximum value $\|\mathbf{x}\|(\alpha^2 + \gamma^2)^{1/2}$. Then an appeal to (6.7) proves (6.1c).

It remains to show how to obtain truncated approximations from a truncator. Suppose that Ξ is a truncator for the linear inverse problem (X, F, G). For any real α, β satisfying $0 < \alpha < \beta$, we define an ordered quintuple $TA = (X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$ as follows:

$$X_{TA} = \Xi(\alpha, \beta), \quad P_{TA} = \Pi_{X_{TA}}, \quad F_{TA} = F|X_{TA}, \quad Q_{TA} = \Pi_{Y_{TA}} \quad G_{TA} = G \tag{6.9}$$

where $Y_{TA} = F(X_{TA})$ and $\Pi_{X_{TA}}$ and $\Pi_{Y_{TA}}$ are the orthogonal projectors of X onto X_{TA} and of Y onto Y_{TA}. We claim that TA is a truncated approximation to (X, F, G). To show this we must prove (3.2). Equations (3.2b) are properties of any orthogonal projectors, and one consequence of (6.1b) is that $F_{TA}^{-1}(\{0\}) = \{0\}$, which proves the existence of the inverse function (3.1a) (Halmos, 1958). That function is continuous because it is linear and its domain is finite dimensional. The functions P_{TA} and Q_{TA} are continuous because they are bounded; indeed, $\|P_{TA}\| = \|Q_{TA}\| = 1$.

In the geomagnetic inverse problem, and perhaps in others, it happens that $\delta_R \mathbf{y} = \delta_R \mathbf{y}_1 + \delta_R \mathbf{y}_2$, where $\delta_R \mathbf{y}_1$ and $\delta_R \mathbf{y}_2$ are independent random errors, and the variance matrix V_1 of $\delta_R \mathbf{y}_1$ is easy to invert while the inverse of V, the full variance matrix of $\delta_R \mathbf{y}$, is obtainable only through heavy computation. In this case it is useful to construct a pre-truncator Π_1 based on V_1. Then (4.6) assures us that Π_1 is also a pre-truncator for V. Thus Π_1 provides a finite-dimensional space on which to carry out numerically the singular value decomposition (6.6) required to find a truncator for the full variance matrix V.

7. PREDICTION ERRORS IN LINEAR INVERSE PROBLEMS WITH DOT PRODUCTS

In this section we examine the error δz defined in (3.7) when the truncated approximation takes the form (6.9). Specifically, we need only the following assumptions about the truncated approximation $(X_{TA}, P_{TA}, F_{TA}, Q_{TA}, G_{TA})$: First, the model space X is a Hilbert space and the prior information about the correct model $\mathbf{x}_E$ is the bound (4.4). Second, the probability measure η_R has been used as in section 4 to provide a dot product on the data space Y. Third, X_{TA} is a finite dimensional subspace of X, and P_{TA} is the orthogonal projector of X onto X_{TA} while Q_{TA} is the orthogonal projector of Y onto $F(X_{TA})$, i.e. onto Y_{TA}.

We define $P_{TA}{}^{\perp}$ and $Q_{TA}{}^{\perp}$ to be the orthogonal projectors of X onto $X_{TA}{}^{\perp}$ and of Y onto $Y_{TA}{}^{\perp}$. Thus

$$P_{TA}{}^{\perp} = P_{X_{TA}{}^{\perp}} : X \to X_{TA}{}^{\perp}, \qquad Q_{TA}{}^{\perp} = P_{Y_{TA}{}^{\perp}} : Y \to Y_{TA} \quad (7.1)$$

Then f_T and g_T in (3.3b) are

$$f_T = -F \circ P_{TA}{}^{\perp}, \qquad g_T = -G \circ P_{TA}^{\perp} \quad (7.2)$$

Since H_T in (3.6b) is linear, we can replace (3.7b) by (3.13) where now $DH_T = H_T$. Thus the $\delta\mathbf{z}$ in (3.7b) is

$$\delta\mathbf{z} = \tilde{\delta}_T\mathbf{z} + \delta_R\mathbf{z} + \delta_S\mathbf{z} \quad \text{where} \quad (7.3a)$$

$$\delta_R\mathbf{z} = H_T(\delta_R\mathbf{y}), \qquad \delta_S\mathbf{z} = H_T(\delta_S\mathbf{y}), \quad \text{and} \quad (7.3b)$$

$$\tilde{\delta}_T\mathbf{z} = (H_T \circ F - G) \circ P_{TA}{}^{\perp}(\mathbf{x}_E). \quad (7.3c)$$

Here $\delta_R\mathbf{z}$ is the random error in predicting $\mathbf{z}_E$ produced by the random error $\delta_R\mathbf{y}$ in the data; $\delta_S\mathbf{z}$ is the systematic error in predicting $\mathbf{z}_E$ produced by the systematic error $\delta_S\mathbf{y}$ in the data; and $\tilde{\delta}_T\mathbf{z}$ is the total truncation error, the systematic error in predicting $\mathbf{z}_E$ produced by the truncated approximation .

First we consider the random error $\delta_R\mathbf{z}$. From (3.14) its probability measure on Z_T is

$$\zeta_R = \eta_R \circ H_T^{-1}. \quad (7.4)$$

The entries of its variance matrix, $E(\delta_R z^i\, \delta_R z^j)$, can be calculated as follows. Let $c^i : Z \to Re$ be the linear functional on Z that assigns to each $\mathbf{z} = (z^1, \cdots, z^P) \in Z$ its i'th entry, z^i, so $c^i(\mathbf{z}) = z^i$. Then $c^i \circ H_T : Y \to Re$ is a linear functional on Y. Since Y is a finite dimensional dot product space, there is a unique vector $\mathbf{H}_T{}^i \in Y$ such that $c^i \circ H_T(\mathbf{y}) = \mathbf{H}_T{}^i \cdot \mathbf{y}$ for all $\mathbf{y} \in Y$ (Halmos, 1958). In particular, then,

$$\delta_R z^i = \mathbf{H}_T{}^i \cdot \delta_R\mathbf{y}. \quad (7.5)$$

Combining (7.5) with (4.8), we conclude that

$$E(\delta_R z^i\, \delta_R z^j) = \mathbf{H}_T{}^i \cdot \mathbf{H}_T{}^j. \quad (7.6)$$

We note that if $\mathbf{y} \in Y_{TA}{}^{\perp}$ then $H_T(\mathbf{y}) = \mathbf{0}$, so $\mathbf{H}_T{}^i \cdot \mathbf{y} = 0$. Therefore $\mathbf{H}_T{}^i \in (Y_{TA}{}^{\perp})^{\perp} = Y_{TA}$. Thus in (7.6) the vectors $\mathbf{H}_T{}^i$ and $\mathbf{H}_T{}^j$ are members of Y_{TA}, a subspace of Y whose dimension is $dim\, Y_{TA}$, i.e. $dim\, X_{TA}$. Usually $D \gg dim\, X_{TA}$, so this fact provides a computational saving in (7.6).

About the systematic error $\delta_S\mathbf{z}$ we can say in general only that its confinement set is $H_T(V_S)$ where V_S is the known confinement set of the systematic error in the data, $\delta_S\mathbf{y}$. If V_S is convex or a convex solid polyhedron or a solid ellipsoid centered on $\mathbf{0}$, so is $H_T(V_S)$. Appendix C shows how to find $H_T(V_S)$ when V_S is an ellipsoid.

The confinement set for the systematic error $\tilde{\delta}_T\mathbf{z}$ is also a solid ellipsoid centered on $\mathbf{0}$, because the confinement set for $\mathbf{x}_E$ is the solid unit ball $B_X(1)$ centered on $\mathbf{0}$ in X (see 4.4). Again appendix C shows how to find that set.

The result that we provide to both subjectivists and objectivists is that with the truncated approximation (6.9) $\mathbf{z}_E$ can be estimated as $\mathbf{z}_0 = H_T(\mathbf{y}_0)$. The total error $\delta\mathbf{z} = \mathbf{z}_0 - \mathbf{z}_E$ in this estimate will be the sum of a random error $\delta_R\mathbf{z}$ and a systematic error $\tilde{\delta}_T\mathbf{z} + \delta_S\mathbf{z}$. The random error has probability measure (7.4) on Z_T, and has variance matrix (7.6). The systematic error has as its confinement set

$$\tilde{\delta}_T\mathbf{z} + \delta_S\mathbf{z} \in (H_T \circ F - G) \circ P_{TA}{}^{\perp}(B_X(1)) + H_T(V_S). \quad (7.7)$$

If V_S is an ellipsoid, so is $H_T(V_S)$, and then (7.7) is the sum of two ellipsoids. This need not be an ellipsoid; the sum of a ball and a needle is a sausage.

At this point objectivists can easily calculate confidence sets for $\delta\mathbf{z}$ at all confidence levels, and will regard the inverse problem as solved. If $dim\, Z \geq 3$, subjectivists can either temporarily become objectivists and accept confidence sets as a solution to the inverse problem, or they can be content to describe $\delta\mathbf{z}$ as the sum of a random error with known probability measure and a systematic error with known confinement set. Appendix A suggests that only if $dim\, Z \leq 2$ will most subjectivists want to exploit their willingness to use a personal probability measure to describe a confinement set. In that case, they gain accuracy by considering $\tilde{\delta}_T\mathbf{z}$ and $\delta_S\mathbf{z}$ separately, rather than combining them as in (7.7). Subjectivists can then separately soften the two hard bounds

$$\tilde{\delta}_T\mathbf{z} \in (G - H_T \circ F) \circ P_{TA}{}^{\perp}(B_X(1)), \qquad \delta_S\mathbf{z} \in H_T(V_S),$$

obtaining two personal probability measures, $\tilde{\zeta}_T$ and ζ_S, for $\tilde{\delta}_T\mathbf{z}$ and $\delta_S\mathbf{z}$ in Z_T. Subjectivists could probably convince each other that $\delta_R\mathbf{z}$, $\delta_S\mathbf{z}$ and $\tilde{\delta}_T\mathbf{z}$ are independent random variables, so they can calculate their personal probability measure for $\delta\mathbf{z}$ in Z_T by convoluting ζ_R, ζ_S and $\tilde{\zeta}_T$. Moreover, they can then calculate the variance matrix of $\delta\mathbf{z}$ as the sum of the variance matrices of $\delta_R\mathbf{z}$, $\delta_S\mathbf{z}$ and $\tilde{\delta}_T\mathbf{z}$.

Equations (6.9) make clear that a single truncator Ξ generates a truncated approximation for every pair (α,β) satisfying (6.1a). Each of these truncated approximations produces different hard and soft bounds for the systematic and random parts of $\delta\mathbf{z}$. Thus, (α,β) can be varied in an attempt to optimize these bounds. If $dim\, Z > 1$, what is optimal will depend on the purpose to which the inversion is put. If $dim\, Z = 1$, then an objectivist inversion will usually produce an interval in Re as the confidence set, and (α,β) can be chosen to minimize the length of this interval. A subjectivist inversion with $dim\, Z = 1$ will produce an interval as the hard bound and a probability measure as the soft bound. The subjectivist might want to choose (α,β) to minimize the sum of the interval half-length and the standard deviation of the probability measure. Thus objectivists and subjectivists will perform the same computations but will interpret them differently.

8. CONCLUSIONS

In geophysical inversion the model space X is usually

infinite dimensional, and the vector space Y of possible data vectors is necessarily finite dimensional. Therefore it is essential to recognize that the observed data vector $\mathbf{y}_0$ cannot provide a complete description of the correct model $\mathbf{x}_E$; $\mathbf{y}_0$ can predict only a finite number of numerical properties of $\mathbf{x}_E$, which we collect into a finite dimensional prediction vector $\mathbf{z}_E$. By focussing on the truncated approximation of the model space and the data function we have constructed a general scheme for solving such inverse problems. It turns out that the truncated approximation must include not only a truncated model space X_{TA} but also two projectors, a model projector P_{TA} that maps X onto X_{TA} and a data projector Q_{TA} that maps Y onto Y_{TA}. Here $Y_{TA} = F(X_{TA})$, and $F: X \to Y$ is the data function for the forward problem.

Our inversion scheme does not require a commitment to either the subjectivist or the objectivist interpretation of probability; both camps can use our results. The scheme does require a careful distinction between random and systematic errors in the data and prediction vectors, and between probabilistic and deterministic prior information about $\mathbf{x}_E$. Concerning the random error $\delta_R\mathbf{y}$ in $\mathbf{y}_0$ we know only that it can be regarded as drawn at random from a population in the data space Y whose probability measure η_R is known. (The question of estimating a few unknown parameters in η_R from the data is discussed by Backus, 1989.) About the systematic error $\delta_S\mathbf{y}$ we know only that $\delta_S\mathbf{y} \in V_S$, a known subset of Y called the confinement set for $\delta_S\mathbf{y}$. There are also truncation errors $\delta_T\mathbf{y}$ and $\delta_T\mathbf{z}$ produced in trying to compute the correct data vector $\mathbf{y}_E$ and the correct prediction vector $\mathbf{z}_E$ from $P_{TA}(\mathbf{x}_E)$, the truncated form of $\mathbf{x}_E$.

The input to our inversion consists of η_R, V_S, $\mathbf{y}_0$ and prior information about $\mathbf{x}_E$. The latter can be a confinement set $U_S \subset X$ for $\mathbf{x}_E$ or a probability measure μ_E describing where $\mathbf{x}_E$ was likely to be in X before the data vector $\mathbf{y}_0$ was measured. The output of the general inversion scheme is an estimate $\mathbf{z}_0$ for $\mathbf{z}_E$ and an expression for the error, $\delta\mathbf{z} = \mathbf{z}_0 - \mathbf{z}_E$, in the form $\delta\mathbf{z} = \Delta\mathbf{z} + \delta_T\mathbf{z}$, where $\Delta\mathbf{z}$ depends on $\delta_R\mathbf{y}$, $\delta_S\mathbf{y}$, $\delta_T\mathbf{y}$ and $\mathbf{y}_0$. In this general form the inversion is of interest mainly to objectivists, because it can be used to find confidence sets for $\delta\mathbf{z}$ but is not easily adapted to untangling the non-linear interaction of random and systematic errors that is of interest to subjectivists. Subjectivists cannot "soften" the confinement sets to probability distributions because almost certainly $dim\ Y$ is so large that such softening introduces spurious extra information.

If the inverse problem and the truncated approximation are linear, $\delta\mathbf{z} = \delta_R\mathbf{z} + \delta_S\mathbf{z} + \widetilde{\delta}_T\mathbf{z}$, where $\delta_R\mathbf{z}$ is the random error produced by $\delta_R\mathbf{y}$, $\delta_S\mathbf{z}$ is the systematic error produced by $\delta_S\mathbf{y}$ and $\widetilde{\delta}_T\mathbf{z}$ is the total truncation error produced by $\delta_T\mathbf{y}$ and $\widetilde{\delta}_T\mathbf{z}$. If the prior information about $\mathbf{x}_E$ is a confinement set, $\widetilde{\delta}_T\mathbf{z}$ is a systematic error. If that prior information is a probability measure for $\mathbf{x}_E$, then $\widetilde{\delta}_T\mathbf{z}$ is a random error. These errors are given a form that makes it easy for objectivists to calculate confidence sets. If $dim\ Z \geq 3$, subjectivists who do not want to introduce spurious "data" through the inversion process may want to leave $\delta\mathbf{z}$ as the sum of a random error with known probability measure and a systematic error with known confinement set. If $dim\ Z \leq 2$, many subjectivists will be willing to "soften" the confinement set to a personal probability distribution and combine the systematic and random parts of $\delta\mathbf{z}$ into a single random error. We conclude that if the systematic errors are comparable to or larger than the random errors it usually makes no sense to try to estimate correlations between predictions. The sole exception is the subjectivist making at most two simultaneous predictions.

We can always introduce on the data space Y a dot product that makes the variance tensor of $\delta_R\mathbf{y}$ equal to the identity tensor on Y. If the prior information about $\mathbf{x}_E$ is a quadratic bound, that bound generates a dot product under which X can be completed to a Hilbert space. When both these dot products are available and the inverse problem is linear, the truncated approximation required for the inversion can be described explicitly. The main computational burden is the singular value decomposition of an $N \times D$ matrix, where N is the dimension of the truncated model space X_{TA} and $D = dim\ Y$. In some inverse problems, the value of D can be very considerably reduced by replacing $\mathbf{y}_0$ with $\widetilde{K}(\mathbf{y}_0)$, where $\widetilde{K}: Y \to \widetilde{Y}$ is a linear "compression function" chosen so that $dim\ \widetilde{Y} \ll D$.

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APPENDIX A: RISKS IN GAUSSIAN BOUND SOFTENING

In spaces of high dimension, "softening" a quadratic bound on a vector $\mathbf{x}$ by imitating the bound with a personal probability measure generates very precise information about $\mathbf{x}$ that comes not from the bound but from the softening process (Backus, 1988b). For spaces of low dimension, the extent of this problem can be studied quantitatively in the special case that the personal probability measure is gaussian. In the present appendix, we will consider a vector $\mathbf{x}$ about which we know only that it belongs to a real dot product space X of dimension $N < \infty$ and that

$$\|\mathbf{x}\| \leq 1. \tag{A.1}$$

When N is not extremely large, what are the consequences of trying to imitate (A.1) with a gaussian personal probability measure on X? How much new "information" does this bound softening generate?

Unless the density $p(\mathbf{x})$ of the personal probability measure is a function of $\|\mathbf{x}\|$ alone, it will single out some directions in X, generating new "information" unjustified by

(A.1). Therefore, since p is gaussian, there must be a constant γ such that

$$p(\mathbf{x}) = (\gamma/2\pi)^{N/2} \exp(-\gamma \| \mathbf{x} \|^2/2). \quad \text{(A.2)}$$

It follows that $\gamma \| \mathbf{x} \|^2$ has a chi-square distribution with N degrees of freedom. The Bayesian literature discusses at length but does not settle how to choose γ (Jackson, 1978; Gubbins, 1983; Backus, 1988a). We will leave γ unspecified since our argument is independent of its value. We will study the random variable

$$s = \gamma \| \mathbf{x} \|^2/2 . \quad \text{(A.3a)}$$

The probability density function for s is (Kendall and Stuart, 1977, p397)

$$p(N,s) = \Gamma(N/2)^{-1} s^{N/2-1} e^{-s} . \quad \text{(A.3b)}$$

The mean and variance of s are both $N/2$, and, for large N, $p(N,s)$ is asymptotically gaussian with that mean and variance. Therefore for large N

$$1-\alpha(2/N)^{1/2} < 2s/N < 1+\alpha(2/N)^{1/2} \quad \text{(A.4)}$$

with probability $\phi = \mathrm{erf}(\alpha/\sqrt{2})$. Thus (A.4) is true at confidence level ϕ. But whenever (A.4) is true, the ratio of the largest to the smallest admissible s is

$$r(N,\phi) = [1+\alpha(2/N)^{1/2}]/[1-\alpha(2/N)^{1/2}]. \quad \text{(A.5)}$$

Since s is a constant multiple of $\| \mathbf{x} \|^2$, at confidence level ϕ we know $\| \mathbf{x} \|^2$ to within the factor $r(N,\phi)$. For fixed ϕ and large N, $r(N,\phi)$ is very close to 1. Therefore, softening (A.1) to (A.2) enables us to extract, apparently from (A.1) and at confidence level ϕ, the actual value of $\| \mathbf{x} \|^2$ with a very small *per cent* error. This paradox is the objection to bound softening when *dim X* is large. Softening the upper bound (A.1) introduces a lower bound for $\| \mathbf{x} \|$ that is generated entirely from the belief that the upper bound can be described probabilistically.

The question remains, whether such a paradox persists when N is not large. To pursue this question, let us suppose that (4.3b) is the probability density for s. Let us fix N, ϕ and s_B, and let us choose $s_T(N,\phi,s_B)$ so that with probability ϕ

$$s_B \le s \le s_T . \quad \text{(A.6)}$$

If (A.6) holds, then we know s to within the factor

$$r(N,\phi,s_B) = s_T(N,\phi,s_B)/s_B . \quad \text{(A.7)}$$

If we minimize $r(N,\phi,s_B)$ with respect to s_B, we obtain the boldest assertion permitted by (A.2) for the given N and ϕ: at confidence level ϕ we know $\| \mathbf{x} \|^2$ to within a factor

$$r(N,\phi) = \min s_T(N,\phi,s_B)/s_B , \quad \text{(A.8)}$$

the minimum being taken with respect to s_B. Table 1 gives $r(N,\phi)$ for various dimensions N and confidence levels ϕ. That table means, for example, that if $N = 30$ then replacing (A.1) by (A.2) and choosing γ provides us with the value of $\| \mathbf{x} \|^2$ to within a factor of 2.5, 3.0 or 5.4 according as we accept a confidence level of 0.9, 0.95 or 0.99. This conclusion is independent of the value we choose for γ.

TABLE 1. The factor $r(N,\phi)$ to within which $\| \mathbf{x} \|^2$ is specified by gaussian softening of a hard quadratic bound, as a function of the dimension N of x and the acceptable confidence level ϕ

N \ ϕ	0.5	0.9	0.95	0.99
1	9.8	520	2500	8400
2	4.4	47	112	760
3	3.2	19.8	38	150
4	2.7	12.4	21	64
5	2.4	9.2	14.5	38
10	1.85	4.6	6.2	11.4
20	1.54	2.9	3.5	5.3
30	1.42	2.4	2.8	3.9
50	1.31	1.94	2.2	2.8
100	1.21	1.59	1.74	2.1
1000	1.06	1.16	1.19	1.26
∞	1	1	1	1

If (A.1) really is all we know about x, purists of the objective school of probability will use table 1 to reject (A.2) as a substitute for (A.1), whatever the value of N. Subjectivists who accept hypotheses verified at a confidence level of 0.95 might be willing to substite (A.2) for (A.1) when $N = 1$ or 2, especially if they could convince themselves that they really were able to estimate $\| \mathbf{x} \|^2$ to within a factor of 2500 or of 112. When $N \ge 3$, softening introduces a lower bound on $\| \mathbf{x} \|^2$ high enough that whether to soften (A.1) to (A.2) begins to be a personal matter. Of course this is not a drawback for subjectivists. In any case, Table 1 will provide each subjectivist with a quantitative basis for deciding whether to soften the hard bound (A.1).

There is one more argument against softening a hard bound when $N \ge 3$. The density function (A.3b) vanishes at $s = 0$ for $N \ge 3$ but not for $N = 1$ or 2. Thus, accepting (A.2) when $N \ge 3$ really does express a disbelief in values of $\| \mathbf{x} \|^2$ considerably less than 1.

APPENDIX B: NOTATION

If U, V and W are sets, $p \in V$ means that p is a member of V, and $U \subset V$ means that U is a subset of V. If $U_1, \cdots, U_N$ are sets then $U_1 \times \cdots \times U_N$, their Cartesian product, is the set of all ordered N-tuples $(u_1, \cdots, u_N)$ with $u_i \in U_i$, $i = 1, \cdots, N$. The set $V \cup W$ consists of all objects belonging to V or W or both. The set $V \cap W$ consists of all objects common to V and W. The set $W \setminus V$ consists of all objects in W and not in V. The set with no members is written ϕ. The set whose only members are $u_1, \cdots, u_N$ is written $\{u_1, \cdots, u_N\}$. The

set whose members are all the subsets of X is written 2^X. If U and V are arbitrary subsets of a vector space Y, then $U+V$ is the set of all vectors in Y that can be written as $\mathbf{u}+\mathbf{v}$ for at least one $\mathbf{u}\in U$ and one $\mathbf{v}\in V$. The set $U-V$ is defined similarly from $\mathbf{u}-\mathbf{v}$. The sets $\{\mathbf{u}\}+V$ and $\{\mathbf{u}\}-V$ are usually abbreviated as $\mathbf{u}+V$ and $\mathbf{u}-V$. If U is a subset of vector space Y then *span* U is the set of all finite linear combinations of members of U; *span* U is a subspace of Y.

If X and Y are sets, a function $f:X\to Y$ is a rule that assigns to each $x\in X$ a unique value $f(x)\in Y$; multivalued functions are disallowed. The sets X and Y are called the domain and codomain of f, and x is the "argument" of f. The notation $f(x)$ always stands for a value of the function f, never for the function itself. The function is sometimes written $f(\cdot)$. Thus, for example, if $f:U\times V\to Y$ and $u_0\in U$ and $v_0\in V$, then $f(u_0,\cdot):V\to Y$ and $f(\cdot,v_0):U\to Y$ are defined as follows: $f(u_0,\cdot)(v)=f(u_0,v)$ for all $v\in V$, and $f(\cdot,v_0)(u)=f(u,v_0)$ for all $u\in U$. If $U\subset X$ then $f(U)$ is the set of all $y\in Y$ that can be written $f(x)$ for at least one $x\in U$. The set $f(X)$ is the "image" of $f:X\to Y$. If $V\subset Y$ then $f^{-1}(V)$ is the set of all $x\in X$ such that $f(x)\in V$. The notation does not require or imply that the inverse function $f^{-1}:Y\to X$ exists. However, the function $f^{-1}:2^Y\to 2^X$ always exists. When f^{-1} exists in both senses, the two meanings of $f^{-1}(V)$ agree. If for each $y\in f(X)$, $f^{-1}(\{y\})$ contains only one member, f is an "injection". If $f(X)=Y$, f is a "surjection." If f is both, it is a "bijection," and then it has an inverse, $f^{-1}:Y\to X$.

If $f:X\to Y$ and $U\subset X$ then $f|U$, the "restriction" of f to U, is that function $g:U\to Y$ such that $g(x)=f(x)$ whenever $x\in U$. If $x\notin U$, $g(x)$ is not defined. If A,B,C are sets and $p:A\to B$ and $q:B\to C$ then the "composite" of p and q, written $q\circ p$, is that function $q\circ p:A\to C$ such that for each $x\in A$, $(q\circ p)(x)=q(p(x))$. Usually $(q\circ p)(x)$ is abbreviated $q\circ p(x)$. If D is another set and $r:C\to D$ is another function, clearly $r\circ(q\circ p)=(r\circ q)\circ p$. If U is any set then the identity function on U, $I_U:U\to U$, is defined by requiring that $I_U(u)=u$ for each $u\in U$. If $f:U\to V$ then clearly $f=I_V\circ f=f\circ I_U$. If $f:U\to V$ has an inverse, $f^{-1}:V\to U$, then $f\circ f^{-1}=I_V$ and $f^{-1}\circ f=I_U$. We denote the set of all real numbers by $\mathbf{R}$. A real-valued function $f:U\to\mathbf{R}$ is a "functional" on U. If Y is a real vector space, X is any set, and $f:X\to Y$, $g:X\to Y$, and $a,b\in\mathbf{R}$, we define the function $(af+bg):X\to Y$ by requiring for each $x\in X$ that

$$(af+bg)(x)=af(x)+bg(x). \tag{B.1}$$

Now we turn to probability measures. If X is any set, a "σ-algebra" on X is an object Σ_X with these properties: i) Σ_X is a set whose members are some of the subsets of X; ii) $X\in\Sigma_X$; iii) if $U,V\in\Sigma_X$ then $U\setminus V\in\Sigma_X$; iv) if $V_i\in\Sigma_X$ for $i=1,2,\cdots$ then $V_1\cup V_2\cup V_3\cup\cdots\in\Sigma_X$. A probability measure on X is a pair (μ,Σ_X) in which Σ_X is a σ-algebra on X and $\mu:\Sigma_X\to\mathbf{R}$. The function μ must have these additional properties: i) $\mu(X)=1$; ii) $\mu(U)\geq 0$ if $U\in\Sigma_X$; iii) μ is countably additive; that is, if $V_1,V_2,\cdots\in\Sigma_X$ and $V_i\cap V_j=\emptyset$ for all $i\neq j$ then $\mu(V_1\cup V_2\cup V_3\cup\cdots)=\sum_{i=1}^{\infty}\mu(V_i)$. The sets belonging to Σ_X are the "μ-measurable" subsets of X. Halmos (1950) gives the theory of integration with respect to measures. If (μ,Σ_X) is a probability measure on X and if the function $f:X\to\mathbf{R}$ is μ-integrable, then its integral is written $\int_X d\mu(x)\,f(x)$. We abbreviate the integral as $E(\mu,f)$ and call it the expected value or mean of f with respect to μ. When there is no danger of confusion, we write simply $E(f)$.

Suppose that Σ_X and Σ_Y are σ-algebras on X and Y, and $F:X\to Y$. We say that F is "measurable" if $F^{-1}(V)\in\Sigma_X$ whenever $V\in\Sigma_Y$. If F is measurable, it defines a function $F^{-1}:\Sigma_Y\to\Sigma_X$ that exists whether or not F has an ordinary inverse $F^{-1}:Y\to X$. If (μ,Σ_X) is a probability measure on X and $F:X\to Y$ is measurable and $\eta=\mu\circ F^{-1}$, then (η,Σ_Y) is a probability measure on Y. For any $V\in\Sigma_Y$, $\eta(V)=\mu(F^{-1}(V))$. The measure η is "induced" on Y by F and μ. Induced measures give a succinct formula for changing variables of integration. If $f:Y\to\mathbf{R}$ then $E(\mu,f\circ F)=E(\eta,f)$, or

$$E(\mu,f\circ F)=E(\mu\circ F^{-1},f). \tag{B.2}$$

In the present paper if Y is a finite-dimensional real vector space, then Σ_Y will always be the set of all Borel subsets of Y; that is, Σ_Y is the smallest σ-algebra that includes as members all open subsets of Y. Therefore, if Y and Z are finite-dimensional real vector spaces and $H:Y\to Z$ is continuous, H is measurable (Halmos, 1950). In particular, if H is linear it is measurable.

A dot product on a vector space X is a symmetric, scalar-valued bilinear form on X. A dot product space is a vector space together with a particular dot product. A real Hilbert space is a real dot product space which is complete in the norm defined by the dot product. The following remarks paraphrase Halmos (1951). For any subset U of a real dot product space X we define $U^{\perp}$, the orthogonal complement of U, to be the set of all $\mathbf{x}\in X$ such that $\mathbf{x}\cdot\mathbf{u}=0$ for every $\mathbf{u}\in U$. If X is a Hilbert space, $U^{\perp}$ is a closed subspace of X. If U is a closed subspace of Hilbert space X then $(U^{\perp})^{\perp}=U$, and for each $\mathbf{x}\in X$ there are unique vectors $\mathbf{u}(\mathbf{x})\in U$ and $\mathbf{u}^{\perp}(\mathbf{x})\in U^{\perp}$ such that $\mathbf{x}=\mathbf{u}(\mathbf{x})+\mathbf{u}^{\perp}(\mathbf{x})$. Then we can define a function $\Pi_U:X\to U$ by setting $\Pi_U(\mathbf{x})=\mathbf{u}(\mathbf{x})$ for all $\mathbf{x}\in X$. This function is the orthogonal projector of X onto U. It is linear, idempotent [i.e. $\Pi_U\circ\Pi_U=\Pi_U$] and symmetric [i.e. $\mathbf{x}_1\cdot\Pi_U(\mathbf{x}_2)=\mathbf{x}_2\cdot\Pi_U(\mathbf{x}_1)$]. Clearly, $\Pi_U\circ\Pi_{U^{\perp}}=\Pi_{U^{\perp}}\circ\Pi_U=0$, and $\Pi_U+\Pi_{U^{\perp}}=I_X$.

If Y and Z are finite-dimensional real dot product spaces and $F:Y\to Z$ is linear, then $F^T:Z\to Y$ is the unique linear

function such that for every $\mathbf{y} \in Y$ and every $\mathbf{z} \in Z$

$$F(\mathbf{y}) \cdot \mathbf{z} = \mathbf{y} \cdot F^T(\mathbf{z}). \tag{B.3}$$

F^T is the "adjoint" or "transpose" of F. Clearly $(F^T)^T = F$. If either F or F^T has an inverse, so does the other, and $(F^T)^{-1} = (F^{-1})^T$. In that case we abbreviate both $(F^T)^{-1}$ and $(F^{-1})^T$ as F^{-T}. If W is another finite-dimensional real dot product space and $G : W \to Y$ is linear, then $(F \circ G)^T = G^T \circ F^T$.

APPENDIX C: VIRIAL BOUND ON MAGNETIC ENERGY

In this appendix our goal is to prove that

$$(2\mu_0)^{-1} \int_{\mathbf{R}^3} dV\, B^2 < (8\pi G)^{-1} \int_{\mathbf{R}^3} dV\, g^2 \tag{C.1}$$

where μ_0 is the permeability of vacuum, $\mathbf{B}$ is the geomagnetic field, G is Newton's universal constant of gravitation, $\mathbf{g}$ is the earth's gravitational acceleration field, and $\mathbf{R}^3$ is all of three-space.

There is an appealing physical argument for (C.1). The left side of that inequality is E_B, the energy of the geomagnetic field, while its right side is $-E_G$ where E_G is the gravitational self energy of the earth. The total energy of the earth is $E_B + E_G + K + U$, where K is kinetic energy (mostly rotational) and U is internal energy. If we imagine the earth to be a perfect electrical conductor, and disassemble it by expanding it uniformly to infinity at constant angular momentum, then each of the four terms in the total energy will tend to 0. The assembled earth presumably has a smaller energy than its disassembled state, and K and U are positive, so we have (C.1).

A more formal proof starts with the pre-Maxwell equations, obtained by neglecting the displacement current. The momentum equation for the matter in the earth is then

$$\rho D_t \mathbf{u} = \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}. \tag{C.2}$$

where ρ is the density of matter, $\mathbf{u}$ is its velocity, $\mathbf{T}$ is its stress tensor, and $\mathbf{J}$ is the electric current density. If ∂_t denotes the time derivative at a point fixed in space, then $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the time derivative at a particle moving with the material. Therefore, if $\mathbf{r}$ is the position vector, then $\mathbf{u} = D_t \mathbf{r}$, from which follows

$$\mathbf{r} \cdot D_t \mathbf{u} = \tfrac{1}{2} D_t^2 r^2 - u^2. \tag{C.3}$$

Another appeal to the pre-Maxwell equations yields the well known result that

$$\mathbf{J} \times \mathbf{B} = \nabla \cdot \mathbf{M} \tag{C.4a}$$

where the Cartesian components of the magnetic stress tensor $\mathbf{M}$ are

$$M^{ij} = \mu_0^{-1} (B^i B^j - \tfrac{1}{2} B^2 \delta^{ij}). \tag{C.4b}$$

It is less well known but equally useful that

$$\rho \mathbf{g} = \nabla \cdot \boldsymbol{G} \tag{C.5a}$$

where the Cartesian components of the gravitational stress tensor $\mathbf{G}$ are

$$G^{ij} = -(4\pi G)^{-1} (g^i g^j - \tfrac{1}{2} g^2 \delta^{ij}). \tag{C.5b}$$

Equations (C.5) follow from $\nabla \cdot \mathbf{g} = -4\pi G \rho$ and $\nabla \times \mathbf{g} = \mathbf{0}$.

We write the Cartesian components of $\mathbf{r}$ as r^i or r_i and those of ∇ as ∂^i or ∂_i, so that we can use the Einstein summation convention. The Cartesian components of (C.2) can now be written

$$\rho D_t u^i = \partial_j Q^{ij} \quad \text{where} \tag{C.6a}$$

$$Q^{ij} = T^{ij} + M^{ij} + G^{ij}. \tag{C.6b}$$

If we multiply (C.6a) by r_i and sum on i, then an appeal to (C.3) and the product rule for derivatives gives

$$\tfrac{1}{2} \rho D_t^2 r^2 - \rho u^2 = \partial_j (r_i Q^{ij}) - \delta_{ij} Q^{ij}. \tag{C.7}$$

We want to integrate (C.7) over all of $\mathbf{R}^3$. If ∂V is the surface of the region V occupied by the earth, we can integrate (C.7) over V and $\mathbf{R}^3 \backslash V$ separately and use Gauss's divergence theorem to cancel the contributions of $\partial_j (r_i Q^{ij})$ from the two sides of ∂V. Thus we obtain

$$\tfrac{1}{2} \int_{\mathbf{R}^3} dV\, \rho D_t^2 r^2 = \int_{\mathbf{R}^3} dV\, \rho u^2 - \int_{\mathbf{R}^3} dV\, \delta_{ij} Q^{ij}. \tag{C.8}$$

For any scalar field f, we can write $\int_{\mathbf{R}^3} dV\, \rho f$ as $\int_{\mathbf{R}^3} dm\, f$ where $dm = \rho\, dV$ is an element of mass. If dm is always the same parcel of mass in the moving material, dm is independent of time, so $(d/dt) \int_{\mathbf{R}^3} dm\, f = \int_{\mathbf{R}^3} dm\, D_t f$. Therefore

$$\frac{d}{dt} \int_{\mathbf{R}^3} dV\, \rho f = \int_{\mathbf{R}^3} dV\, \rho D_t f. \tag{C.9}$$

Throughout most of the earth, $\delta_{ij} T^{ij} = -3p$ to high accuracy, and $\delta_{ij} M^{ij} = -B^2/(2\mu_0)$ and $\delta_{ij} G^{ij} = g^2/(8\pi G)$. Thus, applying (C.9) twice to the left side of (C.8) and noting that p and ρ vanish outside V, we obtain

$$\frac{1}{2} \frac{d^2}{dt^2} \int_V dV\, \rho r^2 = 2K + 3 \int_V dV\, p + E_M + E_G. \tag{C.10}$$

The left side of (C.10) is negligible compared to the pressure integral on the right, even in a large earthquake, and $p > 0$, $K > 0$, so (C.10) implies (C.1).

Equation (C.10) is the scalar virial equation. For other derivations of versions of it, see Chandrasekhar (1961, p577) and Parker (1979, p58).

APPENDIX D: LINEAR IMAGES OF SOLID ELLIPSOIDS

In this appendix we show that the linear image of a solid ellipsoid centered on the origin is another such ellipsoid, and we discuss briefly how to find the image. This image question arises in several contexts when we calculate confinement sets, so we will try to keep the discussion as general as possible.

We begin with a real vector space X about which we know nothing else. Its dimension may be finite or infinite, and it may or may not have a topology, a norm or a dot product. In such an "unfurnished" vector space the simplest definition of a solid ellipsoid centered on $\mathbf{0}$ is this: let $Q: X \times X \to Re$ be any positive definite symmetric (pds) bilinear form on X, just as in (4.1). Let $El(X,Q)$ be the set of all $\mathbf{x} \in X$ such that

$$Q(\mathbf{x},\mathbf{x}) \le 1. \tag{D.1}$$

Any subset of X constructed in this way from a pds bilinear form on X will be called a solid ellipsoid in X centered on $\mathbf{0}$. Given any such ellipsoid, we can use it to furnish X with a dot product, namely

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = Q(\mathbf{x}_1,\mathbf{x}_2). \tag{D.2}$$

Under this dot product and its norm, $El(Q,X)$ is the solid unit ball consisting of all $\mathbf{x} \in X$ such that $\|\mathbf{x}\| \le 1$.

To find the linear images of $El(Q,X)$ we will need the orthogonal projectors of X provided by (D.2) onto the closed subspaces of X. Therefore we ask that X be complete in the norm generated by (D.2). That is, X must be a real Hilbert space with dot product (D.2). If $dim\ X < \infty$, X is automatically complete. If $dim\ X = \infty$, we must assume completeness, and we do so.

Next, suppose that Y is another real vector space, and $dim\ Y < \infty$. Endow Y with the unique topology that makes it a topological linear space (Dunford and Schwartz, 1958, p372), i.e. the ordinary Euclidean topology in which convergence of a sequence of vectors means convergence of all their component sequences relative to each basis for Y. Suppose that $L: X \to Y$ is linear and continuous. (Continuity follows from linearity if $dim X < \infty$.) We will show that there is a pds bilinear form $\tilde{Q}$ on $L(X)$ such that

$$L(El(X,Q)) = El(L(X),\tilde{Q}). \tag{D.3}$$

That is, $L(El(X,Q))$ is a solid ellipsoid centered on $\mathbf{0}$ in $L(X)$.

To find $\tilde{Q}$ let N_L be the null space of L, the set of all $\mathbf{x} \in X$ such that $L(\mathbf{x}) = \mathbf{0}$. Since L is continuous, N_L is a closed subspace of X. Under (D.2), let $P^{\perp}: X \to N_L$ be the orthogonal projector of X onto N_L, and let $P = I_X - P^{\perp}$. Then P is the orthogonal projector of X onto $N_L{}^{\perp}$. Define $K: P(X) \to L(X)$ as

$$K = L \mid P(X). \tag{D.4a}$$

We claim that K has an inverse,

$$K^{-1}: L(X) \to P(X). \tag{D.4b}$$

To prove this, we show that K is both injective and surjective. Since K is linear, to show that it is injective we need show only that $\mathbf{x} \in P(X)$ and $K(\mathbf{x}) = \mathbf{0}$ imply $\mathbf{x} = \mathbf{0}$. But if $\mathbf{x} \in P(X)$ then $\mathbf{x} \in N_L{}^{\perp}$, while if $K(\mathbf{x}) = \mathbf{0}$ then $L(\mathbf{x}) = \mathbf{0}$ so $\mathbf{x} \in N_L$. Since $N_L \cap N_L{}^{\perp} = \{\mathbf{0}\}$, the conclusion follows.

To prove that K is surjective, suppose that $\mathbf{y} \in L(X)$. Then $\mathbf{y} = L(\mathbf{x})$ for some $\mathbf{x} \in X$. Thus $\mathbf{y} = L \circ P(\mathbf{x}) + L \circ P^{\perp}(\mathbf{x})$. Since $P^{\perp}(\mathbf{x}) \in N_L$, therefore $L \circ P^{\perp}(\mathbf{x}) = \mathbf{0}$, and $\mathbf{y} = L \circ P(\mathbf{x})$. But $P(\mathbf{x}) \in P(X)$, so $L \circ P(\mathbf{x}) = K(P(\mathbf{x}))$. Thus $\mathbf{y} \in K(P(X))$, and K is surjective.

Knowing that K^{-1} exists, we can now define $\tilde{Q}: L(X) \times L(X) \to Re$ as

$$\tilde{Q}(\mathbf{y}_1,\mathbf{y}_2) = Q(K^{-1}(\mathbf{y}_1),K^{-1}(\mathbf{y}_2)) \tag{D.5}$$

for all $\mathbf{y}_1, \mathbf{y}_2 \in L(X)$. Clearly $\tilde{Q}$ is a pds bilinear form on $L(X)$. To prove (D.3), first we show that $L(El(X,Q)) \subset El(L(X),\tilde{Q})$. Suppose that $\mathbf{y} \in L(El(X,Q))$. Then $\mathbf{y} = L(\mathbf{x})$ for some $\mathbf{x} \in X$ that satisfies (D.1). But $L(\mathbf{x}) = L \circ P(\mathbf{x}) = K(P(\mathbf{x}))$, so $P(\mathbf{x}) = K^{-1}(\mathbf{y})$. Moreover, $\|P(\mathbf{x})\| \le \|P\|\,\|\mathbf{x}\|$ and $\|P\| \le 1$, $\|\mathbf{x}\| \le 1$, so $\|K^{-1}(\mathbf{y})\| \le 1$, or $Q(K^{-1}(\mathbf{y}),K^{-1}(\mathbf{y})) \le 1$. Hence $\tilde{Q}(\mathbf{y},\mathbf{y}) \le 1$, and $\mathbf{y} \in El(L(X),\tilde{Q})$.

To prove that $El(L(X),\tilde{Q}) \subset L(El(X,Q))$ we suppose that $\mathbf{y} \in El(L(X),\tilde{Q})$. Then $\mathbf{y} \in L(X)$ and $\tilde{Q}(\mathbf{y},\mathbf{y}) \le 1$. Let $\mathbf{x} = K^{-1}(\mathbf{y})$. Then $\mathbf{x} \in P(X)$, and $Q(\mathbf{x},\mathbf{x}) \le 1$. Thus $\mathbf{y} = K(\mathbf{x})$ for an $\mathbf{x} \in L(El(X,Q))$. This completes the proof.

When $dim\ X = \infty$, the foregoing calcualtions require summing infinite series, and cannot be done exactly on a computer. The remedy is to construct a truncated approximation as in sections 3 and 6, as if (X,L,L) were an inverse problem.

REFERENCES

Backus, G.E., Inference from inadequate and inaccurate data, I, *Proc. Nat. Acad. Sci., 65*, 1-7, 1970a.

Backus, G.E., Non-uniqueness of the external geomagnetic field determined by surface intensity measurements, *J. Geophys. Res., 75*, 6339-6341, 1970b.

Backus, G.E., Gross thermodynamics of heat engines in deep interior of earth, *Proc. Nat. Acad. Sci., 72*, 1555-1558, 1975.

Backus, G.E., Bayesian inference in geomagnetism, *Geophys. J., 92*, 125-142, 1988a.

Backus, G.E., Comparing hard and soft bounds in geophysical inverse problems, *Geophys. J., 94*, 249-261, 1988b.

Backus, G.E., Confidence set inference with a prior quadratic bound, *Geophys. J., 97*, 119-150, 1989.

Backus, G.E. and Gilbert, J.F., Numerical applications of a formalism for geophysical inverse problems, *Geophys. J. R. astr. Soc., 13*, 247-276, 1967.

Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, 654pp, Oxford University Press, Oxford, 1961.

Dunford, N. and Schwartz, J.T., *Linear Operators, Part I*, 858pp. Interscience, New York, 1958.

Golub, G.H. and Van Loan, C.F., *Matrix Computations*, 476pp, Johns Hopkins, Baltimore, 1983.

Gubbins, D., Geomagnetic field analysis-I. Stochastic inver-

sion, *Geophys. J. R. astr. Soc., 73*, 641-652, 1983.

Halmos, P.R., *Measure Theory*, 304pp, Van Nostrand, New York, 1950.

Halmos, P.R., *Introduction to Hilbert Space*, 114pp, Chelsea, New York, 1951.

Halmos, P.R., *Finite Dimensional Vector Spaces*, 199pp, Van Nostrand, New York, 1958.

Jackson, D., The use of *a priori* data to resolve non-uniqueness in linear inversion, *Geophys. J. Roy. astr. Soc., 57*, 137-158, 1979.

Jeffreys, H., *Cartesian Tensors*, 92pp, Cambridge University Press, Cambridge, 1969.

Kendall, M. and Stuart, A., *The Advanced Theory of Statistics I*, 472pp, Macmillan, New York, 1977.

Kendall, M. and Stuart, A., *The Advanced Theory of Statistics II*, 748pp, Macmillan, New York, 1979.

MacLane, S. and Birkhoff, G., *Algebra*, 598pp, Macmillan, New York, 1967.

Neyman, J., Outline of a theory of statistical estimation based on the classical theory of probability, *Phil. Trans. R. Soc. A, 236*, 333-380, 1937.

Parker, E.N., *Cosmical Magnetic Fields*, 841pp, Oxford University Press, Oxford, 1979.

Parker, R.L., Inverse theory with grossly inadequate data, *Geophys. J. Roy. astr. Soc., 29*, 123-138, 1972.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P., *Numerical Recipes in C*, 994pp, Cambridge University Press, Cambridge, 1992.

Stark, P., Strict bounds and applications, *Technical Report 147*, 11pp, Department of Statistics, University of California, Berkeley, CA, 1988.

Tarantola, A., *Inverse Problem Theory*, 613pp, Elsevier, New York, 1987.

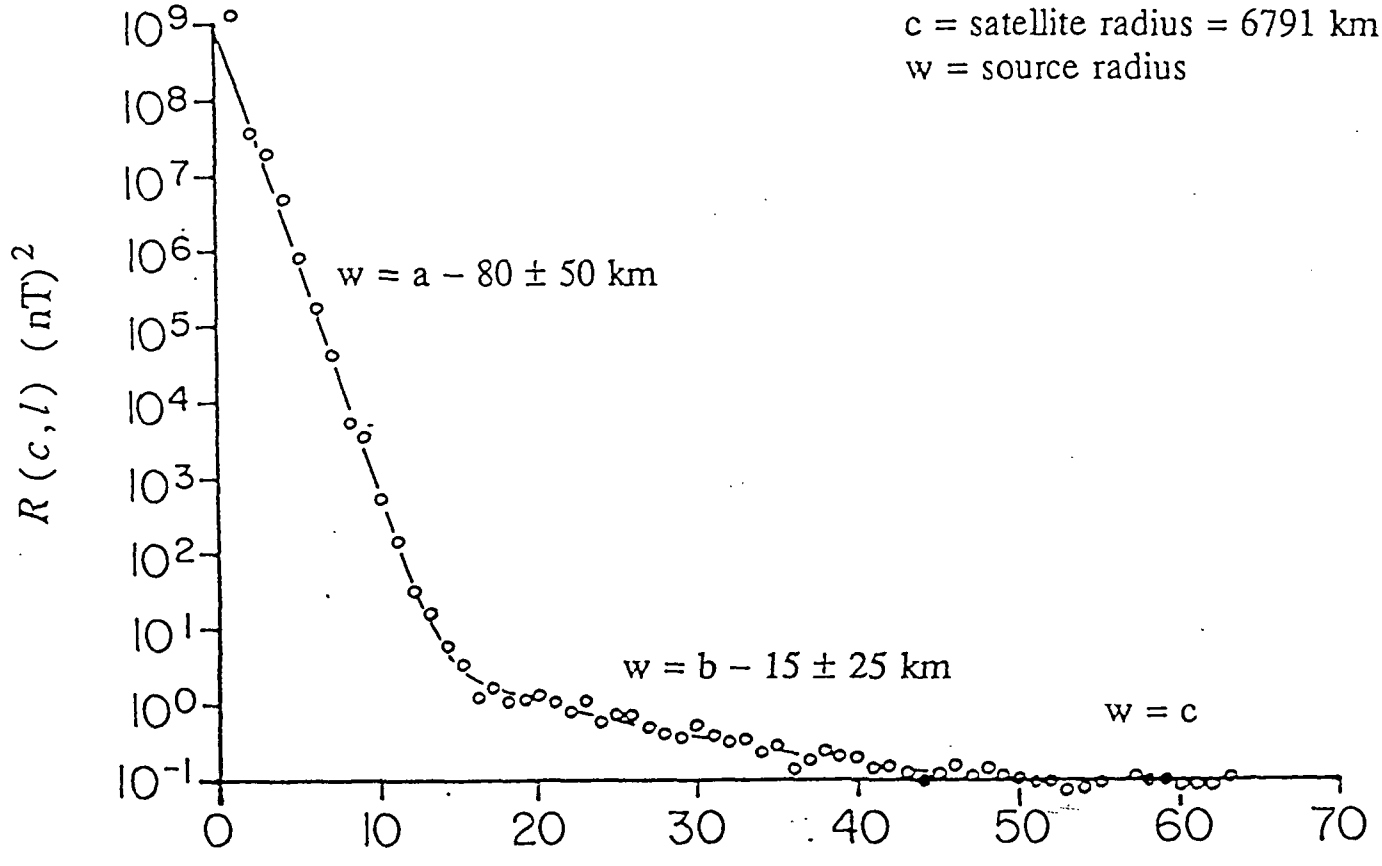
APPENDIX B

Lücke-Mauersberger-Lowes Power Spectrum from Magsat Data

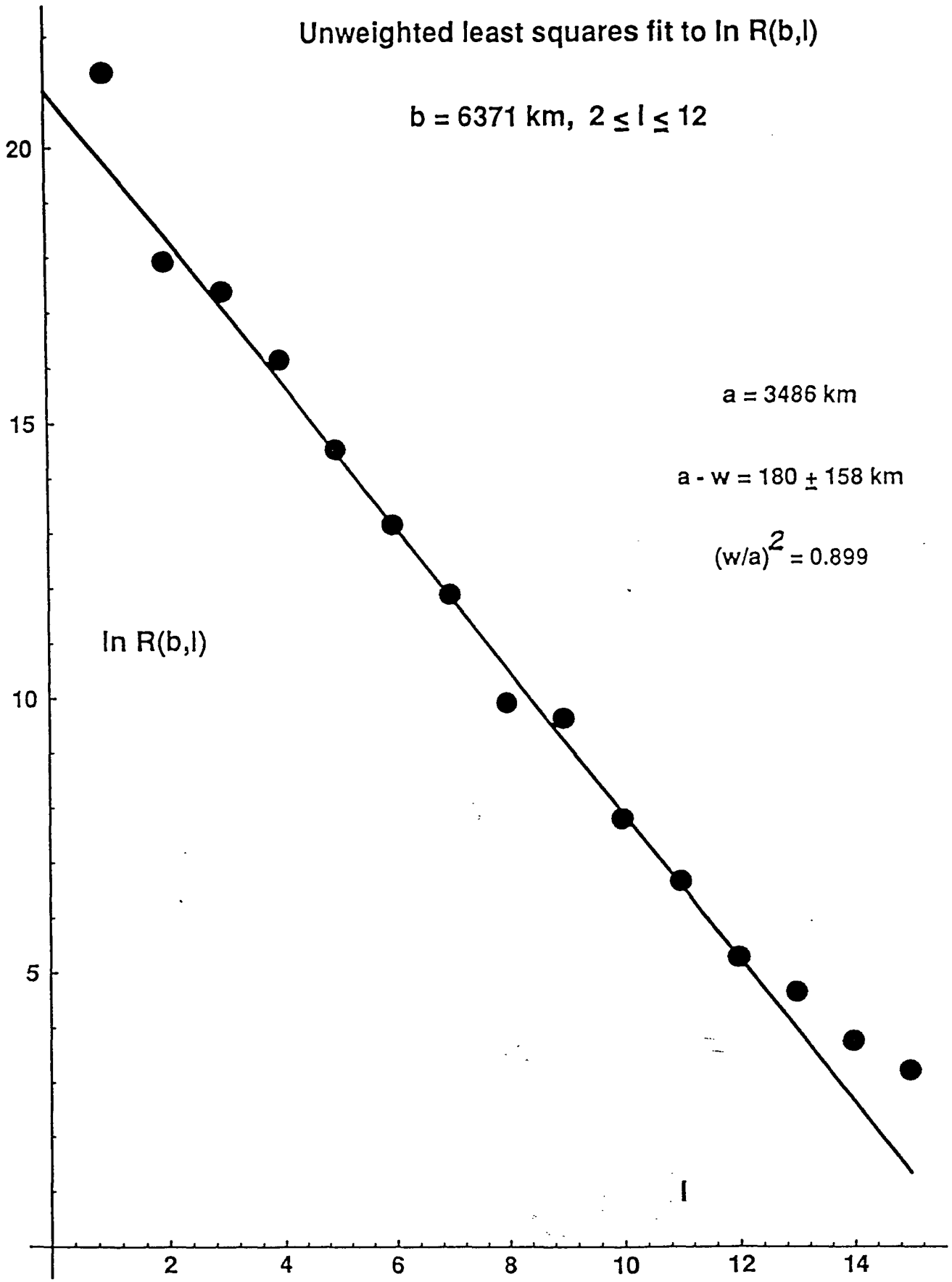
J. Cain, Z. Wang, D. Schmitz & J. Meyer (1989), *Geophysical Journal* 97, 443-447

$$R(c,l) = (l+1)\sum_{m=-l}^{m=l} g_l^m(c)^2 = \frac{1}{4\pi c^2}\int_{S(c)} dA(\mathrm{r})\,|\mathrm{B}_l(\mathrm{r})|^2$$

a = core radius = 3486 km
b = earth radius = 6371 km
c = satellite radius = 6791 km
w = source radius



APPENDIX B


Unweighted least squares fit to ln R(b,l)
b = 6371 km, 2 ≤ l ≤ 12
a = 3486 km
a - w = 180 ± 158 km
(w/a)^2 = 0.899
ln R(b,l)
l
20
15
10
5
2
4
6
8
10
12
14

APPENDIX B

Finding the Source Depth from the Power Spectrum

Origin 0 at center of earth, and position vector $\mathbf{r} = r\hat{\mathbf{r}}$, $|\hat{\mathbf{r}}| = 1$.

$S(b)$ = spherical surface of radius b centered on 0

$\langle f \rangle_{S(b)}$ = area average of f on $S(b)$.

$H_l^m(\mathbf{r})$ = real spherical harmonic polynomial of degree l in $\mathbf{r}$, $-l \le m \le l$.

$$\langle H_l^m H_{l'}^{m'} \rangle_{S(1)} = \delta_{ll'}\,\delta_{mm'} \qquad \text{(i.e. } H_l^m \text{ fully normalized).}$$

If $\mathbf{B}$ = magnetic field produced by the core, and $r > a$ = core radius, then $\mathbf{B} = -\nabla\phi$ where for any b

$$\phi(\mathbf{r}) = b\sum_{l=1}^{\infty}(b/r)^{l+1}\sum_{m=-l}^{m=l} g_l^m(b)\,(2l+1)^{-1/2}\,H_l^m(\hat{\mathbf{r}}).$$

$$b^{l+2}g_l^m(b) = c^{l+2}g_l^m(c) \quad \text{for all } b \text{ and } c.$$

$$\langle |\mathbf{B}_l|^2 \rangle_{S(r)} = (l+1)\sum_{m=-l}^{m=l} g_l^m(r)^2 = R(r,l)$$

= Lücke–Mauersberger–Lowes Spectrum.

$$R(r,l) = (c/r)^{2l+4}R(c,l).$$

$$\langle |\mathbf{B}|^2 \rangle_{S(r)} = \sum_{l=1}^{\infty} R(r,l) = \sum_{l=1}^{\infty}(c/r)^{2l+4}R(c,l).$$

APPENDIX B

Isotropically Random Scalar Fields f on the Spherical Surface S(1)

$$E[f(\hat{\mathbf{r}})] = 0.$$

$$E[f(\hat{\mathbf{r}})\, f(\hat{\mathbf{s}})] = K(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}}) = E[f^2]\, C(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}}).$$

K = autocovariance function of f on $S(1)$

C = autocorrelation function of f on $S(1)$

$$f(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m H_l^m(\hat{\mathbf{r}}).$$

$$E[f_l^m f_{l'}^{m'}] = k_l\, \delta_{ll'}\, \delta_{mm'}, \quad 0 \le l, \ |m| \le l.$$

$$K(\mu) = \sum_{l=0}^{\infty} (2l+1)\, k_l\, P_l(\mu).$$

White Noise

$$K(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}}) = 4\pi\sigma^2\, \delta(\hat{\mathbf{r}} - \hat{\mathbf{s}}).$$

$$E[f_l^m f_{l'}^{m'}] = \sigma^2\, \delta_{ll'}\, \delta_{mm'}, \quad 0 \le l, \ |m| \le l.$$

$$k_l = \sigma^2, \qquad 0 \le l$$

APPENDIX B

White Noise Non-dipole Radial Field B_r on S(w)

All Sources Inside S(w), and r > w

$$B_r(r\,\hat{\mathbf{r}}) = \sum_{l=2}^{\infty} \sum_{m=-l}^{m=l} \gamma_l^m(r)\, H_l^m(\hat{\mathbf{r}}).$$

$$r^{l+2}\gamma_l^m(r) = w^{l+2}\gamma_l^m(w) \quad \text{if} \quad r > w.$$

$$E[\gamma_l^m(w)\,\gamma_{l'}^{m'}(w)] = \sigma^2 \delta_{ll'}\delta_{mm'}, \quad 2 \le l, \; -l \le m \le l.$$

$$\boxed{E[\gamma_l^m(r)\,\gamma_{l'}^{m'}(r)] = \sigma^2 (w/r)^{2l+4} \delta_{ll'}\delta_{mm'}, \quad 2 \le l, \; -l \le m \le l.}$$

$$\gamma_l^m = (l+1)(2l+1)^{-1/2} g_l^m.$$

$$\tilde{\gamma}_l^m = (l+1)^{1/2}(2l+1)^{1/2} g_l^m = (l+1)^{-1/2}(2l+1)\,\gamma_l^m.$$

C. Constable and R. Parker, *JGR* **93**, 11569–11581 (1988)
V. Courtillot, J-P. Valet, G. Hulot & J-L. LeMouël, *EOS*, **73**, 337–342 (1992)

$$f(w\,\hat{\mathbf{r}}) = -(4\pi)^{-1} \int_{S(1)} dA(\hat{\mathbf{s}})\, Q(\hat{\mathbf{r}}\cdot\hat{\mathbf{s}})\, \nabla_1^2 B_r(w\,\hat{\mathbf{s}}) \qquad \text{where}$$

$$Q(\mu) = \sum_{l=2}^{\infty} (2l+1)(l+1)^{-3/2} P_l(\mu) \qquad \text{and}$$

$$\nabla_1^2 = r^2\nabla^2 - r\,\partial_r^2 r \; = \text{angular part of laplacian}$$

APPENDIX B

Expected l dependence of L-M-L spectrum R(c, l) if B_r is white noise on S(w)

$$g_l^m(c) = (2l+1)^{1/2}\,(l+1)^{-1}\,\gamma_l^m(c)$$

$$\gamma_l^m(c) = (w/c)^{l+2}\,\gamma_l^m(w)$$

$$E[\gamma_l^m(w)\,\gamma_{l'}^{m'}(w)] = \sigma^2\,\delta_{ll'}\,\delta_{mm'}$$

$$R(c,l) = (l+1)\sum_{m=-l}^{m=l} g_l^m(c)^2$$

$$\ln R(c,l) + \ln(l+1) - \ln(2l+1) - \zeta_l = \\ = l\,\ln[(w/c)^2] + \ln[2(w/c)^4\sigma^2]$$

$$\zeta_l = \ln \tfrac{1}{2} \sum_{m=-l}^{m=l} [\gamma_l^m(w)/\sigma]^2$$

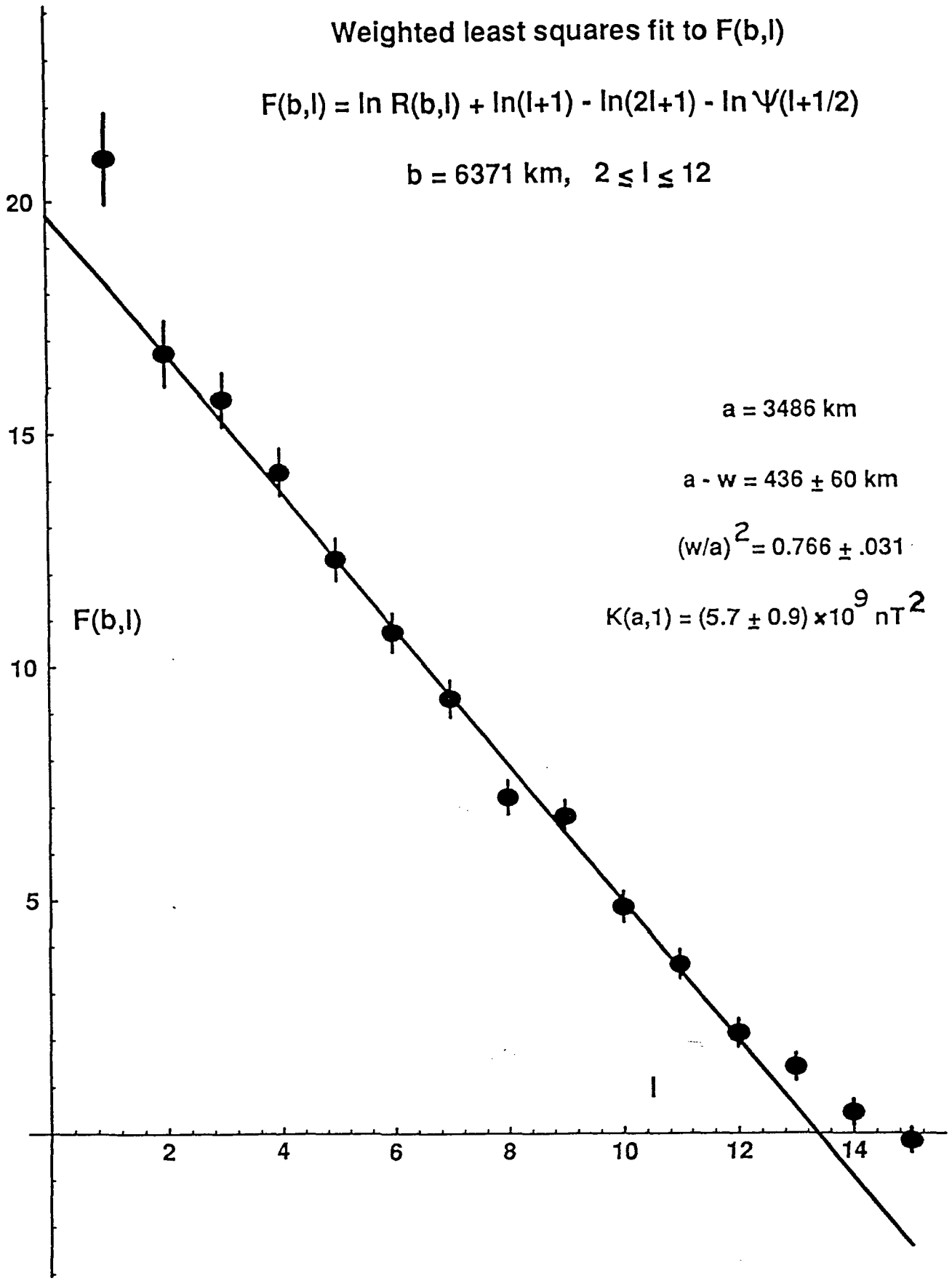
$$E[\ln R(c,l)] + \ln(l+1) - \ln(2l+1) - E[\zeta_l] = \\ = l\,\ln[(w/c)^2] + \ln[2(w/c)^4\sigma^2]$$

$$E[\zeta_l] = \psi(l+\tfrac{1}{2}) \quad \approx \ln l \quad \text{as} \quad l \to \infty$$

$$Var[\ln R(c,l)] = Var[\zeta_l] = \partial_z \psi(l+\tfrac{1}{2})$$

$$\psi(z) = \partial_z \ln \Gamma(z)$$

APPENDIX B


Weighted least squares fit to F(b,l)
F(b,l) = ln R(b,l) + ln(l+1) - ln(2l+1) - ln Ψ(l+1/2)
b = 6371 km, 2 ≤ l ≤ 12
a = 3486 km
a - w = 436 ± 60 km
(w/a)^2 = 0.766 ± .031
K(a,1) = (5.7 ± 0.9) x10^9 nT^2
F(b,l)
l
20
15
10
5
2
4
6
8
10
12
14

APPENDIX B

Two-Parameter Stochastic Model of Non-Dipole Radial Field on CMB

S(a) = core-mantle boundary, a = 3486 km
S(c) = satellite orbital sphere, c = 6791 km

$$B_r(r\hat{\mathbf{r}}) = \sum_{l=2}^{\infty} \sum_{m=-l}^{m=l} \gamma_l^m(r) H_l^m(\hat{\mathbf{r}}).$$

$$r^{l+2}\gamma_l^m(r) = c^{l+2}\gamma_l^m(c).$$

General Isotropic Stochastic Model

$$E[B_r(r\hat{\mathbf{r}})B_r(r\hat{\mathbf{s}})] = K(r, \hat{\mathbf{r}}\cdot\hat{\mathbf{s}})$$

$$E[\gamma_l^m(r)\ \gamma_{l'}^{m'}(r)] = k_l(r)\,\delta_{ll'}\,\delta_{mm'}$$

$$K(r,\mu) = \sum_{l=2}^{\infty}(2l+1)\ k_l(r)\ P_l(\mu)$$

$$r^{2l+4}k_l(r) = c^{2l+4}k_l(c)$$

White Noise Model: $k_l(w) = \sigma^2$ for some radius w

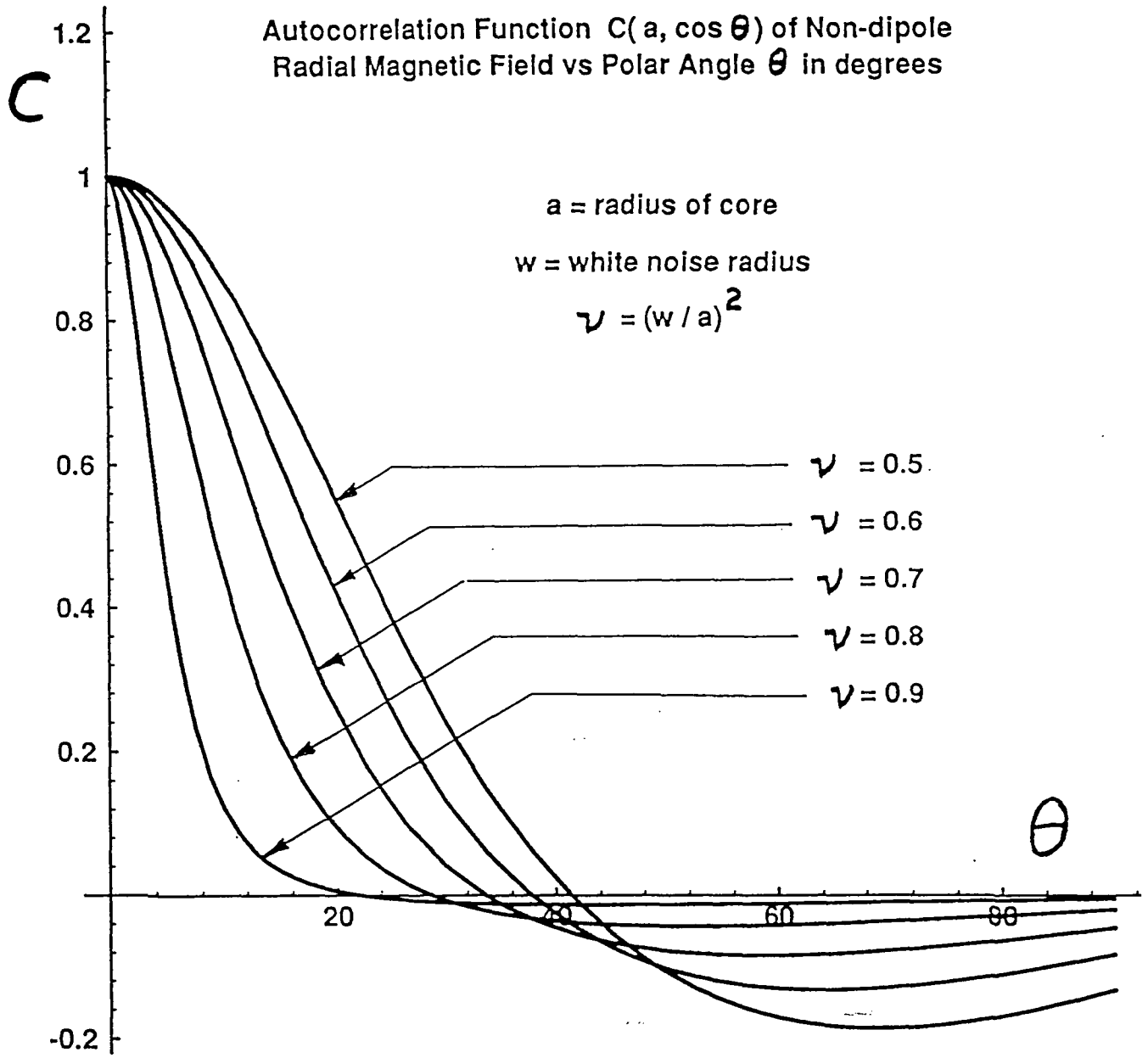
$$\nu = (w/a)^2$$

$$K(a,\mu) = \sigma^2 \sum_{l=2}^{\infty}(2l+1)\ \nu^{l+2}\ P_l(\mu) = K(a,1)\,C(a,\mu)$$

$$\nu = 0.766 \pm 0.031$$

$$K(a,1) = (6 \pm 3)\times 10^{10}\ \ \mathrm{nT}^2$$

APPENDIX B


Autocorrelation Function C(a, cos θ) of Non-dipole
Radial Magnetic Field vs Polar Angle θ in degrees
C
1.2
1
0.8
0.6
0.4
0.2
-0.2
a = radius of core
w = white noise radius
ν = (w / a)²
ν = 0.5
ν = 0.6
ν = 0.7
ν = 0.8
ν = 0.9
θ
20
60
90

APPENDIX B

CONCLUSIONS

1. The MAGSAT data for $2 \le l \le 12$ are well fitted by an isotropic stochastic model with two adjustable parameters: a power level σ^2 and an apparent white noise radius w.

2. In these models, the l dependence of $E[\ln R(b,l)]$ includes a digamma function $\psi(l+\frac{1}{2})$ and some logarithmic terms as well as a term linear in l. Also, $Var[\ln R(b,l)] = \partial_z \psi(l+\frac{1}{2})$, independent of b, σ^2, or w.

3. The white noise radius w depends on which scalar field is modeled as white noise on $S(w)$. For B_r, w is about 440 km below the CMB. This corresponds to a correlation length (half peak width at half power) of about 12 degrees or 750 km.

$$\text{b} = \text{earth radius}, \qquad \mathbf{B} = -\nabla\phi$$

$$\phi(\mathbf{r}) = b \sum_{l=2}^{\infty} (b/r)^{l+1} \sum_{m=-l}^{m=l} g_l^m(b)\,(2l+1)^{-1/2}\, H_l^m(\hat{\mathbf{r}}).$$

$$R(b,l) = (l+1) \sum_{m=-l}^{m=l} g_l^m(b)^2$$

$$\psi(z) = \partial_z \ln\Gamma(z)$$